Mathematical Reviews

Edited by

J. L. Doob **W. S. Massey** **J. V. Wehausen**

S. H. Gould, ***Executive Editor***

Chandler Davis, W. Freiberger ***and*** **J. A. Zilber,** ***Associate Editors***

H. A. Pogorzelski, ***Copy Editor***

Vol. 21, No. 1 **JANUARY, 1960** **Reviews 1–644**

TABLE OF CONTENTS

AUTHOR INDEX

(*Continued on cover 3*)

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4 (82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place.

Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, 190 Hope St., Providence 6, R.I.

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DE FRANCE
DANSK MATEMATISK FORENING
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
POLSKIE TOWARZYSTWO MATEMATYCZNE
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY
UNIONE MATEMATICA ITALIANA
THE SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

Editorial Office

MATHEMATICAL REVIEWS, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to individual members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Printed in Great Britain by William Clowes and Sons, Limited, London and Beccles

Mathematical Reviews

Vol. 21, No. 1 January, 1960 Reviews 1–644

LOGIC AND FOUNDATIONS

See also 620, 640.

1:

★**Synge, J. L. Kandelman's Krim: A realistic fantasy.** Jonathan Cape, London, 1957. 175 pp. 15s.

This is an unusual book. With the possible exception of Lewis Carroll's *Sylvie and Bruno* (1889–93) or C. H. Hinton's *Scientific Romances* (1922–25), nothing at all like it has appeared since *Through the Looking-Glass.* Instead of Humpty-Dumpty, the White Knight, and Alice, we meet the Orc (an agreeable monster who seems to speak for the author himself), the Unicorn ("It is a good life. I eat grass, and flowers too when I find them, and think about mathematics. What more could a unicorn want?"), the Kea (a fierce parrot who will not tolerate any nonsense), the Goddess (quick at numerical computation), and the practical Plumber. They are evidently not English but Irish. With their minds clarified by a potion of Kandelman's Krim, they converse about irrational numbers, complex numbers, and infinity. The Orc, whose home is in the ocean, is free from such arbitrary habits as counting in tens. When the Plumber asks him for the square root of two, he begins to work it out in the binary scale, with a highly entertaining result. The mathematics in this book, though simple, is all "good" mathematics; every student will find it not only enjoyable but helpful. The story, which includes some remarkable climaxes, can be enjoyed as a good story or interpreted allegorically in various ways to suit the reader's taste. Whether it will appeal, like the "Alice" books, to children as well as adults, remains to be seen.

The preface is as freely discursive as Bernard Shaw's prefaces to his plays. It deals in part with the "operative negative" (the principle that every meaningful statement must have a meaningful denial) as applied to the question of the existence of God. It includes biographical sketches of two mathematical bishops: George Berkeley (1685–1753), who stated that reality is to be found only in our sense-perceptions, and Hugh Hamilton (1729–1805), who resisted the "modern" tendency by writing a synthetic treatise on conics. The latter bishop (who was apparently not related to the great Hamilton) is described more fully in the appendix. There also the author's ancestry is traced back through twelve generations (including Bishop Hugh) to Sir James Hamilton of Fynnart in Scotland, who was executed in 1540 on a charge of high treason.

H. S. M. Coxeter (Toronto, Ont.)

2:

Šanin, N. A. A constructive interpretation of mathematical judgments. Trudy Mat. Inst. Steklov. **52** (1958), 226–311. (Russian)

The aim of the paper is to give a constructive interpretation of existential propositions on the basis of Markov's theory of normal algorithms. There are given alphabets $A_0, \cdots, A_h$, where $A_0=\{0, 1\}$ serves for denotation of natural numbers. The subscript i will everywhere take the values $0, \cdots, h$. The words in these alphabets are the objects of the theory, of types $0, \cdots, h$ respectively. The first step consists in adjoining signs (among them $\square$, ι, κ) which allow the denotation of normal algorithms over A_i; moreover, any such algorithm can be described by a word in A_0. The denotation of an algorithm is called a functor. $\langle H\rangle_i$ is an abbreviated notation for the functor associated with the algorithm with description H. There are also adjoined variables for objects and for functors. Let us call this language $\mathfrak{T}_1$. The notion of a term can now be defined in a natural way; each term is either an object-term or a functor-term. For an object-term containing no variable, a process of valuation can be defined in an obvious way. An object-term makes sense if the process of valuation comes to an end. If Θ and Δ are object-terms, then $!\Theta$ and $\Theta\simeq\Delta$ are elementary formulas, with the intended interpretation "Θ makes sense" and "if either Θ or Δ makes sense, then Θ and Δ have the same value". Formulas are built up out of elementary formulas by the signs of the first order predicate calculus. A normal formula is a formula which contains neither $\vee$ nor $\exists$; a completely regular formula is of the form $\exists\alpha_1\cdots\alpha_n F$, where F is a normal formula.

A reduction process is described by which every formula is reduced to a completely regular formula. Thus the interpretation of a formula consists in the problem of constructing certain words satisfying certain conditions. The method of reduction differs from what could be expected from a finitist point of view, because $\exists\alpha A\supset\exists\gamma C$ is replaced by $\exists\gamma\forall\alpha(A\supset C)$. Algorithms C, D_i, E_i, G_i are now described. C selects the natural numbers (by transforming them into the empty word), D_i selects the words in A_i. Let $\langle X\rangle_i$ be a functor, Y a word in $A_i\cup\{\square, \iota, \kappa\}$, and Z a natural number. $\ni$ is a new sign. $G_i(X\ni Y\ni Z)=\Lambda$ if and only if the application of the algorithm $\langle X\rangle_i$ to the word Y ends after at most Z steps. In this case $E_i(X\ni Y\ni Z)=\langle X\rangle_i(Y)$; in the other case, $E_i(X\ni Y\ni Z)$ is the result of the first Z steps in the application of $\langle X\rangle_i$ to Y. In the next extension of the language, signs C, D_i, E_i, G_i are introduced, for which the intended interpretation consists in the algorithms described above. In the extended language $\mathfrak{T}_2$ a notion of terms is introduced, which, in order to distinguish them from the terms in $\mathfrak{T}_1$, are called auxiliary terms. Elementary auxiliary formulas are of the form $\Psi_1=\Psi_2$, where Ψ_1 and Ψ_2 are auxiliary terms; auxiliary formulas are built up from them by the signs of the first order predicate calculus. The definition of normal and of completely regular auxiliary formulas are analogous to those of the corresponding sorts of formulas.

$\mathfrak{T}_2$ is used for the interpretation of $!\Theta$ and of $\Theta\simeq\Delta$. If $\langle K\rangle_i$ is a constant functor and Y a word in $A_i\cup\{\square\}$, then $!\langle K\rangle_i(Y)$ is interpreted by

$$\neg\forall\alpha((C(\alpha) = \Lambda) \supset \neg(G_i(K \ni Y \ni \alpha) = \Lambda)),$$

which means that the process of application of $\langle K\rangle_i(Y)$ is not infinite. This is an application of Markov's rule, which in its general form comes to the adjunction of the formula

$$\forall x(A \vee \neg A) \,\&\, \neg\forall x \neg A \to \exists x A$$

to the intuitionistic predicate calculus. Markov's rule contains a second deviation from the finitist point of view: The interpretation of $!\Theta$ and $\Theta \simeq \Delta$ for arbitrary terms can now be given by normal auxiliary formulas. Some results on the interpretation of general auxiliary terms are given without proof. For constant elementary auxiliary formulas a truth definition is given. The author says that the extension of this truth definition to general auxiliary formulas can be effected by methods which are analogous to Lorenzen's, though he does not consider Lorenzen's work [*Einführung in die operative Logik und Mathematik*, Springer, Berlin–Gottingen–Heidelberg, 1955; MR **17**, 223] as definitively established.

If we use only those axioms of the predicate calculus [in the form of Kleene, *Introduction to metamathematics*, van Nostrand, New York, 1952; MR **14**, 525], in which $\vee$ and $\exists$ do not occur, every constant normal auxiliary formula which is derivable is true. The same holds for formulas with $\vee$ and $\exists$, if $\mathfrak{A} \vee \mathfrak{B}$ is replaced by $\neg(\neg\mathfrak{A} \,\&\, \neg\mathfrak{B})$ and $\exists x\, \mathfrak{A}$ by $\neg\forall x \neg\mathfrak{A}$.

The last part of the paper contains applications to set theory and the introduction of restricted quantifiers. It is asserted that the set defined by the formula $\mathfrak{B}(x)$ is recursive if and only if the formula $\forall x(\mathfrak{B}(x) \vee \neg\mathfrak{B}(x))$ is true.

A. Heyting (Amsterdam)

3:

Moh, Shaw-Kwei. Modal systems with a finite number of modalities. Acta Math. Sinica **7** (1957), 1–27. (Chinese. English summary)

If we add the modal concepts "$\Diamond p$" (it is possible that p), "$\sim\Diamond p$" (it is impossible that p), and "$\Box p$" (it is necessary that p) etc., to the truth-valued logical system, we get a modal logical system. A proposition which is formed by means of propositional variables and the two connectors "$\sim$" "$\Box$" alone is called a modality.

As to the five modal systems proposed by C. I. Lewis, $S1, \cdots, S5$, it has been proved that the systems $S1$ and $S2$ contain infinitely many modalities and that the systems $S3$, $S4$ and $S5$ contain only a finite number of modalities, the number of which are 42, 14 and 6 respectively [Parry, J. Symbolic Logic **4** (1939), 137–154; MR **1**, 131; McKinsey, ibid. **5** (1940), 110–112; MR **2**, 66].

In the present paper we start with a basic modal system B, far weaker than the system $S2$, and show that if we add to it the following propositions (where "$\Box^n$" means n symbols "$\Box$" written successively)

$$\sim\Box^3 p \prec \sim\Box^2 p, \quad \sim\Box\sim\Box\sim\Box p \prec \sim\Box p$$

then the resulting system B_1 will contain no more than 126 modalities; and if we add still the proposition

$$\sim\Box^2\sim\Box p \prec \sim\Box^2\sim\Box^2 p$$

then the resulting system will contain exactly the same modalities as that contained in $S3$ (see § 2). We then generalize this result and show that if we add to the system B the following propositions

$$\sim\Box^{n+1} p \prec \sim\Box^n p, \quad \sim\Box^{n-1}\sim\Box\sim\Box p \prec \sim\Box p$$

then the resulting system B_n will likewise contain only a finite number of modalities, no more than (see § 3)

$$2+4\left[(3n^2-1n+1)\sum_{k=0}^{n-1}\binom{n}{k}\binom{n-1}{k}-\sum_{k=1}^{n-1}\sum_{h=k}^{n-1} h\binom{n}{k}\binom{h-1}{k-1}\right].$$

We show that the first proposition is necessary for a system to contain only a finite number of modalities, in a next paper we shall discuss the necessary and sufficient conditions and the corresponding modal systems.

What we have assumed in the deduction is very weak, hence the whole discussion and the results may be written in a purely mathematical language.

Author's summary

4:

Moh, Shaw-kwei. Fundamental finite modal systems. Acta Math. Sinica **8** (1958), 153–180. (Chinese. English summary)

A finite sequence of non-negative integers is called a modal sequence if any of its three consecutive terms of the form "$a0b$" is identified as a single term "$a+b$", for example, 13045 is identified as 175.

For any two modal sequences $\alpha = a_1a_2\cdots a_h$ and $\beta = b_1b_2\cdots b_k$, if we can get the same sequence by replacing some (or no) even terms a_{2i} and some (or no) odd terms b_{2j+1} by smaller integers, then we say that α precedes β, denoted by $\alpha \prec \beta$. It is easy to see that the relation $\prec$ is an order relation.

If $\alpha \prec \beta$ and $\beta \prec \alpha$, then we say that α is equivalent to β, denoted by $\alpha = \beta$.

Evidently we have infinitely many non-equivalent modal sequences. If we assume some new order relations, it may happen that in the resulting systems only a finite number of non-equivalent modal sequences can exist. Such systems will be called finite modal systems. In any finite modal system there are some properties which may be characterized by non-negative numbers (called parameters). When the values of parameters are given, we may deduce some order relations in the system. Such order relations are called essential relations, which hold necessarily, in some sense, in every finite modal systems. For example, the relations $n = n+1$ and $\alpha^h = \alpha^{h+k}$ are essential relations, where α^h denotes $\alpha\alpha\cdots\alpha$ (h in number). The number n is called the order of the system. The number pair (h, k) is called the type of the sequence α. Among the various types of all the sequences of the system, the strongest ones are called the types of the system.

The present paper is to construct finite modal systems when the parameters alone are given, in other words, to construct the finite modal systems from the essential relations alone. Such systems constructed are called fundamental finite modal systems.

The main results are as follows:

We may construct the systems of order 0 and of order 1 readily. They contain 2 and 14 modal sequences, respectively.

When the type $(h, 2k)$ of the sequence 1 and the type $(r_i, 1)$ of the sequence 21^{2i-1} ($i = 1, 2, \cdots, m$, where m is the integral part of $h/2$) are given, we may construct the system of order two. The total number of the modal sequences in it may be computed.

We may construct the systems of the type (1, 2), and the systems of order 3 with type (2, 2). In such construction, however, the sequences which have the type (1, 2) or (2, 2) cannot be given arbitrarily.

By means of zu-sequences (characterized by $z \dashv u$, $z^3=z^5$, and $zuz=zuz^3=z^3uz$) we may construct infinitely many fundamental finite modal systems of order three. Finally, we prove that (under a very broad assumption) if we can construct all the fundamental finite modal systems of order three with type (2, 2), then we can construct all the fundamental finite modal systems.

Author's summary

5:

Detlovs, V. K. Equivalence of normal algorithms and recursive functions. Trudy Mat. Inst. Steklov. **52** (1958), 75–139. (Russian)

A function $\varphi(x_1, \cdots, x_n)$ is called algorithmic if there is a normal algorithm $\mathfrak{F}$ over the alphabet $\{|, *\}$ such that $\varphi(x_1, \cdots, x_n) \simeq \mathfrak{F}(x_1* \cdots *x_n)$. (Here x_i stands for a natural number and at the same time for the word consisting of x_i strokes.) It is completely algorithmic if $\mathfrak{F}$ is applicable to every word in $\{|, *\}$. The following theorems are proved. (I) Every partial recursive function is algorithmic. (II) Conversely. (III) Every recursive function is completely algorithmic. (IV) Conversely. The proof of I is by direct construction of $\mathfrak{F}$. II is first proved for $n=1$; the general case follows easily. For $n=1$, $\mathfrak{F}$ is over $\{|\}$; by a result of Nagornyĭ [same Trudy **52** (1958), 66–74; MR **20** #6356] $\mathfrak{F}$ is then equivalent to an algorithm in $A=\{|, \alpha\}$. Instead of Gödel numbering, a representation γ of words in A by two-rowed matrices is used:

$$\gamma(|) = \mathbf{A} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}; \qquad \gamma(\alpha) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = \mathbf{B};$$

if $\gamma(\xi_i)=\mathbf{Z}_i$ $(i=1, \cdots, n, \ \xi_i \in A)$, then $\gamma(\xi_1 \cdots \xi_n)= \mathbf{Z}_1 \times \cdots \times \mathbf{Z}_n$, where $\times$ denotes matrix multiplication. This gives a one-to-one mapping of the words in A onto the products of matrices $\mathbf{A}$ and $\mathbf{B}$. In this arithmetization the representation of substitution is simpler than by Gödel numbering.

Let M, N, L be words in A; $\mathbf{M}$, $\mathbf{N}$, $\mathbf{L}$ the corresponding matrices; and $\Sigma(M, N, L)$ the result of substitution of L for the first occurrence of N in M; a recursive function σ is constructed such that $\sigma(\mathbf{M}, \mathbf{N}, \mathbf{L}, i)$ $(i=1, \cdots, 4)$ are the elements of $\gamma(\Sigma(M, N, L))$. An arithmetization of an alphabet A is given by two algorithms $\mathfrak{G}$ and $\mathfrak{H}$, of which $\mathfrak{G}$ transforms every word in A into a number and different words into different numbers, while $\mathfrak{H}\mathfrak{G}V=V$. An algorithm $\mathfrak{R}$ over A is called recursive if for some arithmetization of A there is a recursive function φ so that, for every V in A, $\varphi(\mathfrak{G}(V)) \simeq \mathfrak{G}(\mathfrak{R}(V))$. It is proved that every recursive algorithm is equivalent to a normal algorithm, and conversely.

Some of the results of this paper were announced in Dokl. Akad. Nauk SSSR **90** (1953), 723–725 [MR **16**, 436].

A. Heyting (Amsterdam)

6:

Nagornyĭ, N. M. Some generalizations of the concept of a normal algorithm. Trudy Mat. Inst. Steklov. **52** (1958), 1–65. (Russian)

The notion of a normal algorithm was introduced by A. A. Markov [*Teoriya algorifmov*, Trudy Mat. Inst. Steklov., no. 42, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **17**, 1038]. The author introduces three generalizations, algorithms of types σ, σ', σ''. An algorithm of type σ differs from a normal algorithm in the following respect. Instead of one scheme of substitution-formulas a numbered list of such schemes $S_1, \cdots, S_n$ is given. In the application of the algorithm the formula which is applied at the nth step also determines the number of the scheme which must be applied at the $(n+1)$th step. Every algorithm of type σ is equivalent to a normal algorithm; the proof for the theorem on composition of algorithms is much simpler for algorithms of type σ. In an algorithm of type σ' the list of schemes is infinite; it is generated by a normal algorithm. In an algorithm of type σ'', moreover, the formulas in each scheme are generated by a normal algorithm. It is proved that every algorithm of type σ' and even of type σ'' is equivalent to a normal algorithm.

A. Heyting (Amsterdam)

7:

Myhill, J. Recursive equivalence types and combinatorial functions. Bull. Amer. Math. Soc. **64** (1958), 373–376.

Let a collection of numbers (i.e., non-negative integers) be called a set and a collection of sets be called a class. ε, o, Q, V stand for: the set of all numbers, the empty set of numbers, the class of all finite sets, the class of all sets, respectively. The cardinality of the set α is denoted by card α. The sets α and β are recursively equivalent $(\alpha \simeq \beta)$ if there is a partial recursive one-to-one function which maps α onto β. Recursive equivalence types (RETs) are defined as the equivalence classes into which V is decomposed by the $\simeq$ relation. Req α stands for RET represented by α. Since two finite sets are recursively equivalent if and only if they are equivalent, Req o and Req $\{1, \cdots, n\}$ are identified with 0 and n, respectively. The operations

$$\text{Req}\,\alpha + \text{Req}\,\beta = \text{Req}\,(\{2n|n \in \alpha\} \cup \{2n+1|n \in \beta\}),$$

$$\text{Req}\,\alpha \cdot \text{Req}\,\beta = \text{Req}\,\{2^m \cdot 3^n|m \in \alpha \ \&\ n \in \beta\}$$

are well-defined, associative and commutative; multiplication is distributive over addition. Ω denotes the collection of all RETs and

$$\Lambda = \{X|X \in \Omega \ \&\ X \neq X+1\}.$$

Ω and Λ have cardinality c and are closed under addition and multiplication. $A \leqq B$ means $A+X=B$ for some $X \in \Omega$. For the arithmetic of Λ see the reviewer's papers [Math. Z. **70** (1958), 113–124, 250–262; MR **20** #5133, #5134].

Let $\{\rho_n\}$ be an effective enumeration without repetitions of Q such that $r_n = \text{card}\,\rho_n$ is a recursive function of n. A mapping ϕ from V into V is called combinatorial if (1) $\alpha \in Q \Rightarrow \phi(\alpha) \in Q$, (2) card $\alpha = $ card $\beta \Rightarrow$ card $\phi(\alpha) = $ card $\phi(\beta)$, (3) ϕ has a quasi-inverse ϕ^{-1}, i.e., a mapping from $\phi(\varepsilon)$ into Q such that $x \in \phi(\sigma) \Rightarrow \phi^{-1}(x) \subset \sigma$. If ϕ satisfies (1) and (2), the function $f_\phi(n)$ such that f_ϕ (card α) = card $\phi(\alpha)$, for $\alpha \in Q$, is called the function induced by ϕ; $f(x)$ is combinatorial if it is induced by some combinatorial mapping. The functions $x!$, x^x are combinatorial; so are x^k, k^x, $C_{x,k}$ for every fixed k. If $f(x)$ and $g(x)$ are combinatorial, so are $f(x)+g(x)$, $f(x)\cdot g(x)$ and $f(g(x))$. Theorem: $f(x)$ is combinatorial if and only if $f(x)=\sum_{i=0}^{\infty} c_i C_{x,i}$ for some sequence $\{c_i\}$ of numbers; this sequence is uniquely determined by $f(x)$. Moreover, f is also induced by the combinatorial mapping

$$\phi_f(\alpha) = \{2^m \cdot 3^n|\rho_m \subset \alpha \ \&\ n < c_{r(m)}\},$$

which has the property $\alpha \simeq \beta \Rightarrow \phi_f(\alpha) \simeq \phi_f(\beta)$. The function

$f(x)$ can therefore be extended to the function $F(\mathrm{Req}\ \alpha) = \mathrm{Req}\ \phi_f(\alpha)$ from Ω into Ω; F is called the canonical extension of f. A recursive combinatorial (r.c.) function on Ω is defined as the canonical extension of some r.c. function on ε. The functions $A!$, A^A are r.c. on Ω; so are A^k, k^A, $C_{A,k}$ for every fixed k.

We mention some of the theorems (no proofs are supplied). Let $F(A)$ and $G(A)$ be r.c. on Ω. Then: (1) $A \leqq B \Rightarrow F(A) \leqq F(B)$, and if F is not constant, there is a number n_F such that for $X, Y \in \Lambda$,

$$X, Y \geqq n_F \quad \& \quad X \neq Y \Rightarrow F(X) \neq F(Y);$$

(2) if $F(n) = G(n)$ for almost all n, $F(A) = G(A)$ for all $A \in \Omega - \varepsilon$; (3) if $F(n) \neq G(n)$ for almost all n, $F(X) \neq G(X)$ for all $X \in \Lambda - \varepsilon$. It follows from (1) that for $X, Y \in \Lambda - \varepsilon$, $k \geqq 2$, (a) $X^k = Y^k \Rightarrow X = Y$, (b) $k^X = k^Y \Rightarrow X = Y$, (c) $X! = Y! \Rightarrow X = Y$. The notion of a combinatorial function and some of the theorems are generalized to functions of any finite number of variables. Erratum: p. 373 last line, read: addition, multiplication and exponentiation.

J. C. E. Dekker (New Brunswick, N.J.)

8:

Korolyuk, V. S.; and Letičevskiĭ, O. A. A class of address algorithms. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 116–119. (Ukrainian. Russian and English summaries)

This paper deals with the equivalence of algorithms in the class of address algorithms, the description of which utilizes only the operations "segregation of the contents of the address" and "carry-over by address". An algorithm is constructed for the checking of the equivalence of two given address algorithms. *Authors' summary*

9:

Wang, Hao. Eighty years of foundational studies. Dialectica **12** (1958), 466–497. (German and French summaries)

This is a survey of different trends in work on the foundation of mathematics. Following Bernays, the author distinguishes (1) anthropologism or strict finitism, (2) finitism in the sense of Hilbert and Bernays, (3) intuitionism, (4) predicativism or the theory of natural numbers as being, (5) platonism or classical set theory. Of the suggestions on new methods and problems I mention the following: Formalization of finitist methods, including restricted transfinite induction; extension of the notion of recursive ordinal by relativization; derivation of the Skolem paradox for an arbitrary formal system of platonistic set theory. *A. Heyting* (Amsterdam)

ORDER, LATTICES

See also 4, 336, 337.

10:

Felscher, Walter. Beziehungen zwischen verbandsähnlichen Algebren und geregelten Mengen. Math. Ann. **135** (1958), 369–387.

If R is a binary relation on a set E then E is said to be ruled by R. Let A, B be sets, let α map A into B, let β map B into A. Let $<$ denote a partial ordering both in A and in B. A pair (α, β) of mappings, as above, is called a Galois correspondence if (for a, $a_i \in A$; b, $b_i \in B$) $a_1 < a_2$ implies $\alpha a_1 < \alpha a_2$; $b_1 < b_2$ implies $\beta b_1 < \beta b_2$, $a < \beta\alpha a$; $\alpha\beta b < b$. If $\cap$ and $\cup$ are mappings with domain in $E \times E$ and range in E, then the triple $(E, \cap, \cup)$ is called a (lattice-similar) algebra. The author's object is a study of characterizations and embeddings of certain classes of ruled sets. This is effected by defining, between certain classes $\mathscr{A}$ of algebras and the class $\mathscr{B}$ of antisymmetric ruled sets, a correspondence in the following way. (It is understood that there is a fixed underlying set E for both the algebras and ruled sets.) Let $\alpha A \in \mathscr{B}$ be that ruled set which satisfies $y < x$ if and only if (x, y) is in the domain of $\cup$ and $x \cup y = x$; for $B \in \mathscr{A}$, the algebra βB is defined to be the one satisfying: (x, y) is in the domain of $\cup$ if and only if $\sup(x, y)$ exists in B and $\sup(x, y) = x \cup y$, and dually for $\cap$. Among other results there are characterizations of the transitive ruled sets, the reflexive ruled sets, lattices, and distributive lattices. *R. M. Baer* (Berkeley, Calif.)

11:

Popruženko, J. Sur la vitesse de la croissance des suites infinies d'entiers positifs. I. Echelle des vitesses. Fund. Math. **46** (1959), 235–242.

Let $\mathscr{M}$ be the space of all sequences of positive integers which approach ∞. A partial ordering is defined in $\mathscr{M}$ in the following way: if $s = \{n_i\}$ and $t = \{m_i\}$ are elements of $\mathscr{M}$ then $s \gg t$ if $\lim (n_i/m_i) = \infty$. Several properties of this partial ordering of $\mathscr{M}$ are proved, among them the following one: If N is a subset of $\mathscr{M}$ and N is at most denumerable then there exist elements p and q in $\mathscr{M}$ such that $p \gg x \gg q$ for all $x \in N$.

The principal result of the paper is based on this last property as well as on some general theorems, also proved in the paper, concerning well-ordered subsets of a partially ordered set. This principal result states that the propositions (T) and (U) given below are equivalent under the assumption that the cardinal number of the continuum is a regular aleph, that is, that ω_c, the initial ordinal number of the continuum, is a regular ordinal number. (T) There exists in $\mathscr{M}$ a subset E of type $\omega_c^* + \omega_c$ such that, to every $m \in \mathscr{M}$, two elements s and t of E can be found with $t \gg m \gg s$. (U) Every transfinite sequence of elements of $\mathscr{M}$ which is well-ordered and unbounded under the relation $\gg$ is of order type ω_c; and the same holds for the order relation $\ll$. This equivalence of (T) and (U) can also be proved (with ω_c replaced by ω_ν) for any subset of $\mathscr{M}$, provided again that its cardinal number $\aleph_\nu$ is a regular aleph. *F. Herzog* (East Lansing, Mich.)

12:

Brainerd, B.; and Lambek, J. On the ring of quotients of a Boolean ring. Canad. Math. Bull. **2** (1959), 25–29.

If R is a subring of the Boolean ring S, the ring S is called a completion of R if (1) every subset of S has an infimum relative to the natural partial ordering in S, and (2) every element of S is the supremum of some subset of R. It is proved that S is a completion of R if and only if S is the maximal ring of quotients of R. This leads to another construction of the completion of a Boolean ring.

R. E. Johnson (Northampton, Mass.)

13:

Keedy, M. L. On a theorem of Jónsson and Tarski. Portugal. Math. **16** (1957), 11–14.

The theorem in question is theorem 4.30 of Jónsson and Tarski [Amer. J. Math. **73** (1952), 126–162; MR **13**, 524],

which gives a necessary and sufficient condition that an abstract relation algebra be isomorphic to a suitable algebra of binary relations defined in a non-empty set. The condition is a Boolean identity involving atoms. The author of this note gives an equivalent condition involving the representation of atoms in terms of a restricted class of the atoms. *F. B. Wright* (New Orleans, La.)

THEORY OF NUMBERS

See also 87.

14:

Jacobsthal, Ernst. Über einige Eigenschaften der primen Restklassen mod n. Norske Vid. Selsk. Forh. Trondheim **31** (1958), no. 16, 6 pp.

Let $\{a_i\}$, $i=1, 2, \cdots, \phi(n)$, be relatively prime to n and incongruent (mod n). Several additive properties of the a's are proved in a quite elementary way. Sample theorems are: (1) The congruence $a_i+a_j\equiv m \pmod n$ is solvable for every m if n is odd and for every even m if n is even; (2) The numbers $\{a_i+m\}$ are congruent (mod n) in some order to $\{a_i\}$ if and only if all prime factors of n divide m. *L. Moser* (Edmonton, Alta.)

15:

Babaev, G. Remark on a paper of Davenport and Heilbronn. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 63–64. (Russian)

Let $k\geqq 2$ be an integer, and denote by S_k the sequence of numbers n representable in the form $n=p+x^k$, where p is a prime and x is a natural number. It was shown by N. P. Romanoff [Math. Ann. **109**, 668–678 (1934)] that S_k has positive density, and by H. Davenport and H. Heilbronn [Proc. London Math. Soc. (2) **43** (1937), 142–151] that S_k contains almost all integers. In the present note the author uses an easy and elementary argument to establish a result in the opposite direction; he proves, in fact, that there exist infinitely many integers not contained in S_k. *L. Mirsky* (Sheffield)

16:

Nicol, C. A. Linear congruences and the Von Sterneck function. Duke Math. J. **26** (1959), 193–197.

Let $f(m, n, r, s)$ denote the number of solutions of the congruence

$$y_1+\cdots+y_r\equiv s \pmod n,$$

where each of the y_i belongs to the least non-negative residue system modulo n and no integer is repeated more than $(m-1)$ times in any one congruence. The author's principal result is that $f(m, n, r, s)$ is the coefficient of $z^{(n-1)(m-1)-r}$ in the polynomial

$$n^{-1}\sum_{d|n}(z^{\{md,n\}/d}-1)^{mdn/\{md,n\}}/(z^{n/d}-1)^d\Phi(s, n/d),$$

where $\{md, n\}$ is the L.C.M. of md and n and Φ is the Von Sterneck function (also known as Ramanujan's sum) given by

$$\Phi(k, n) = \sum_{\substack{r=1\\(r,n=1)}}^{n} \exp(2\pi irk/n).$$

T. M. Apostol (Pasadena, Calif.)

17:

Mordell, L. J. Integer quotients of products of factorials. J. London Math. Soc. **34** (1959), 134–138.

Lower case Latin letters in this review represent non-negative integers. Erdös [Amer. Math. Monthly **54** (1947), 286] proposed the problem to show that for every c there exist infinitely many x such that $(2x)!/x!(x+c)!$ is an integer. Using an elementary argument, the author generalizes this result as follows: Given a prime p and a, b, c such that $p=al+bm$, where $l>0$ and $m>0$; if $a\neq 0$ or p and if $b\neq p$, then there exist infinitely many x such that $Q=(px)!/[(ax)!]^l[(bx+c)!]^m$ is an integer. Among the special cases not covered by this theorem, the only one of interest is $a=0$, $m=p$, in which case the quotient of factorials becomes $Q=(px)!/[(x+c)!]^p$. It remains an open question whether or not for every c there exist infinitely many x such that Q is an integer. However, the particular case $c=1$ is settled in the affirmative for every integer $p>1$. *T. M. Apostol* (Pasadena, Calif.)

18:

Wright, E. M. A generalization of a result of Mordell's. J. London Math. Soc. **33** (1958), 476–478.

Lower case Latin letters in this review represent non-negative integers. One of the results of a paper by Mordell [reviewed above] is extended as follows: Given $a_1, a_2, \cdots, a_k$, $c_1, c_2, \cdots, c_k$, $k\geqq 2$, and a prime p such that $p=a_1+\cdots+a_k$. If $c_1=0$, $a_1>0$ and $a_2>0$, then there exist infinitely many x such that $(px)!(\prod_{i=1}^k\{(a_ix+c_i)!\})^{-1}$ is an integer. *T. M. Apostol* (Pasadena, Calif.)

19:

Jarden, Dov. Supplementary remarks to the paper: Linear forms of primitive prime divisors of Fibonacci numbers. Riveon Lematematika **12** (1958), 31–32. (Hebrew)

20:

Xeroudakes, George; and Moessner, Alfred. On equal sums of like powers. Proc. Indian Acad. Sci. Sect. A. **48** (1958), 245–255.

21:

McCarthy, Paul J. Odd perfect numbers. Scripta Math. **23** (1957), 43–47 (1958).

This paper is a history of the problem of the existence of odd perfect numbers, in particular during the last twenty years. A list of all papers on this subject since the publication of Dickson's *History of the theory of numbers* is added. (The later paper of the author reviewed below should be added to this list.) *A. Brauer* (Chapel Hill, N.C.)

22:

McCarthy, Paul J. Note on perfect and multiply perfect numbers. Portugal. Math. **16** (1957), 19–21.

It is well known that if an odd perfect number n exists, its prime factorization must have the form

$$n = p^\alpha q_1^{2\beta_1}q_2^{2\beta_2}\cdots q_t^{2\beta_t},$$

where $p\equiv 1\equiv\alpha \pmod 4$. Let $\tau(n)$ be the number of divisors of n and r the smallest odd prime divisor of $\tau(n)$. The author proves that n is not perfect if there exists a prime factor s of n for which $r>s$ and $p+1\not\equiv 0 \pmod s$. In particular, n is not perfect if $r>p$. Moreover, using a

method of O. Grün [Math. Z. **55** (1952), 353–354; MR **14**, 724], a criterion for multiply perfect numbers is obtained.
A. Brauer (Chapel Hill, N.C.)

23:
Kolberg, O. Identities involving the partition functions $q(n)$ and $q_0(n)$. Math. Scand. **6** (1958), 80–86.

Let $q(n)$ denote the number of partitions of n into distinct parts, $q_0(n)$ the number of partitions of n into distinct odd parts. In the same way that he derived identities for $p(n)$, the number of unrestricted partitions of n [Math. Scand. **5** (1957), 77–92; MR **19**, 838], the author proves six identities for $q(n)$, $q_0(n)$, of which the following pair are typical:

$$\sum q(5n)x^n \sum q(5n+2)x^n = (\sum q(5n+1)x^n)^2;$$
$$\sum q_0(5n+1)x^n \sum q_0(5n+7)x^n = (\sum q_0(5n+4)x^n)^2.$$

The methods of proof are elementary and elegant, depending on the dissection of a power series according to the residue class of the exponent modulo a prime.
M. Newman (Washington, D.C.)

24:
Erdös, P. Remarks on number theory. I. On primitive α-abundant numbers. Acta Arith. **5** (1958), 25–33 (1959).

Let $\sigma(n)$ denote the sum of the divisors of n. Davenport [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **26/29** (1933), 830–837], Behrend and S. Chowla proved that $\sigma(n)/n$ has a continuous distribution function. We call m primitive α-abundant if $\sigma(m)/m \geqq \alpha$ but $\sigma(d)/d < \alpha$ for every proper divisor d of m, and denote by $N_\alpha(x)$ the number of primitive α-abundant numbers not exceeding x. The author proves that

$$N_\alpha(x) = o\left(\frac{x}{\log x}\right),$$

which is best-possible in a certain sense.
S. Chowla (Boulder, Colo.)

25:
Bateman, P. T. Remark on a recent note on linear forms. Amer. Math. Monthly **65** (1958), 517–518.

Generalizing a result of A. Brauer for the case $d=1$ [Amer. J. Math. **64** (1942), 299–312; MR **3**, 270], J. B. Roberts [Proc. Amer. Math. Soc. **7** (1956), 456–469; MR **19**, 1038] determined the largest integer N for which the Diophantine equation

$$N = ax_0+(a+d)x_1+\cdots+(a+sd)x_s$$

has no solution in non-negative integers. The author gives a simpler proof for this result. (See also the review of Roberts' paper.) *A. Brauer* (Chapel Hill, N.C.)

26:
Battaglia, Antonio. L'equazione indeterminata $x^{2n}+y^{2n}=z^3$ e l'ultimo teorema di Fermat. Archimede **10** (1958), 120–125.

It is shown that the equation $x^{2n}+y^{2n}=z^3$ has no integral solutions x, y, z subject to the conditions that n is an odd prime and $(z,n)=(x,y)=1$. In the reviewer's opinion, the derivations of equations (21)–(24) requires more justification than is given. *R. A. Rankin* (Glasgow)

27:
Gabard, E. Factorisations et équation de Pell. Mathesis **67** (1958), 218–220.

The first two sections of this note deal with the factors of certain large numbers which were stated by Kraitchik to be almost certainly prime, but which in fact are composite. The third section is concerned with the Pell equation, $y^2-18x^2=1$, whose solutions are given by the recursion formula $y_{n+2}=34y_{n+1}-y_n$. The solutions y_{2m+1}, $m\geqq 1$, were shown by Thébault to be composite, but the author has verified that $y_1=17$, $y_2=577$, $y_4=665837$ are prime and that y_8 and y_{16} are composite. He conjectures that the number of solutions of $y^2-18x^2=1$, where x and y are prime, is finite, and that y_1, y_2, y_4 are the only such solutions. *R. D. James* (Vancouver, B.C.)

28:
Golubev, V. A. Nombres de Mersenne et caractères du nombre 2. Mathesis **67** (1958), 257–262.

This note lists certain criteria for determining whether numbers of the form 2^p-1 are composite: (I) If $8n+7$ is prime it divides $2^{4n+3}-1$; (II) if $p=12n+7$ is prime and of the form x^2+27y^2, it divides $2^{12n+1}-1$ if $12n+1$ is a prime; (III) if $p=72n+31$ is prime and of the form x^2+27y^2, it divides $2^{12n+5}-1$ if $12n+5$ is a prime; (IV) if p and p_1 are prime and $p_1=8p+1=x^2+64y^2$, with y odd, then 2^p-1 is divisible by p_1. The first criterion is Euler's, and the second and third are attributed to Kraitchik and Pellet, but the author states that he obtained them independently in 1937, and exhibits a table of values for x, y, p, p_1. The fourth criterion is due to Storchi [Boll. Un. Mat. Ital. (3) **10** (1955), 363–375; MR **17**, 127], but again the author gives a table of values that he calculated in 1938.

Other divisibility properties of Mersenne numbers are also stated. The final paragraph of the note asserts that there are serious reasons for supposing that the number of Mersenne numbers which are prime is finite.
R. D. James (Vancouver, B.C.)

29:
Sierpiński, W. Sur les nombres premiers de la forme n^n+1. Enseignement Math. (2) **4** (1958), 211–212.

The author proves the following result: Among the numbers having at most 300,000 digits, there are only three, 2, 5, and 257, which are primes of the form n^n+1. (He conjectures that there are no other such numbers.) The proof consists in showing that if n^n+1, $n>1$, is prime then $n=2^{2^m}$ so that n^n+1 is the Fermat number F_{m+2^m}. Since F_1 and F_3 are prime, but F_6 and F_{11} are composite, it follows that, if n^n+1 is a prime for $n>4$, it must exceed F_{20}. *R. D. James* (Vancouver, B.C.)

30:
Erdös, P. Asymptotic formulas for some arithmetic functions. Canad. Math. Bull. **1** (1958), 149–153.

Let α be an irrational number, and let $d(n, m)$ denote the number of divisors of the greatest common divisor of n and m. The author proves that $\sum_{n=1}^{x} d(n, [\alpha n]) \sim \frac{1}{6}\pi^2 x$ if and only if, for every positive number c, there are only a finite number of pairs of positive integers a, b for which $\alpha < a/b < \alpha+(1+c)^{-b}$. He states, without proof, the corresponding result with the number of divisors replaced by the sum of the divisors. *T. Estermann* (London)

31:

Lavrik, A. F. On a theorem in the additive theory of numbers. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 197–198. (Russian)

The problem of representation of integers as sums of primes and of powers of a given integer has been considered, among others, by N. P. Romanoff [Math. Ann. **109** (1934), 668–678] and by Yu. V. Linnik [Mat. Sb. (N.S.) **32** (74) (1953), 3–60; MR **15**, 602]. In the context of these investigations, the following result has been conjectured. Given any integer $g \geqq 2$, there exists an integer $k=k(g)$ such that the density of the sequence of numbers n representable in the form

$$n = p+g^{x_1}+\cdots+g^{x_k}$$

(where p is a prime and the x_i are positive integers) is equal to $\frac{1}{2}$. The author demonstrates that, at any rate for $g>2$, this conjecture is false. Its truth, or otherwise, for $g=2$ remains an open question. *L. Mirsky* (Sheffield)

32:

Malyšev, A. V. Asymptotic distribution of points with integral coordinates on certain ellipsoids. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 457–500. (Russian)

The author considers the Diophantine equation

$$f(x, g, z) = m,$$

where f is a positive ternary quadratic form with integral coefficients and m is a large positive integer. It is supposed that the invariants Ω, Δ (in the classical notation) satisfy $\Delta=1$, Ω odd, and further that $(-f/p)=1$ for each prime factor p of Ω. (Thus f is rationally related to $x^2+y^2+z^2$.) Let $\mathscr{C}$ be a cone in x, y, z space, with vertex at the origin, and with solid angle λ in the space of X, Y, Z, where $f(x, y, z)=X^2+Y^2+Z^2$. Let g be an odd integer satisfying $\Omega g\neq 1$. Let $t(f, \mathscr{C}, g, m)$ denote the number of primitive representations of m by f which lie in $\mathscr{C}$ and for which x, y, z have fixed residues to the modulus g. Then, as $m\to\infty$,

$$t(f, \mathscr{C}, g, m) \sim (\lambda/4\pi)2^k\Omega^{-1}g^{-2}t(m) \prod_{p|\Omega g} (1+p^{-1})^{-1},$$

where $t(m)$ is the number of primitive representations of m as a sum of three squares. The proof makes use of the analytic arithmetic of quaternions, and is an extension and refinement of work by Linnik.

H. Davenport (Cambridge, England)

33:

Malyšev, A. V. Representation of large numbers by positive ternary quadratic forms of odd, relatively prime, invariants. Dokl. Akad. Nauk SSSR. (N.S.) **118** (1958), 1078–1080. (Russian)

The author announces without proof five theorems somewhat similar to the main theorem of his previous paper [reviewed above], but the restrictions on the form f are relaxed considerably. It is now supposed only that Ω, Δ are odd and relatively prime. Now, however, the asymptotic formula is replaced by estimates. The following (theorem 3) may be quoted as representative. Let q be a prime not dividing 2Δ, let g be relatively prime to $2\Omega\Delta$, and let $\mathscr{C}$ be a cone of solid angle λ as before. Let m be a large positive integer, relatively prime to $2\Omega\Delta qg$, such that the congruence $f(x_0, y_0, z_0)\equiv m \pmod{8\Omega\Delta g}$ is soluble and such that $(-\Delta m/q)=1$. Then, with $t(f, \mathscr{C}, g, m)$ as before, there exist m_0, $k>0$, $k'>0$, depending only on Ω, Δ, q, g, $\mathscr{C}$, such that

$$kh(-\Delta m) < t(f, \mathscr{C}, g, m) < k'h(-\Delta m)$$

for $m\geqq m_0$, where $h(-\Delta m)$ denotes the number of classes of positive, properly primitive, binary forms of determinant Δm. *H. Davenport* (Cambridge, England)

34:

Linnik, Yu. V. Dispersion of divisors and quadratic forms in progressions and certain binary additive problems. Dokl. Akad. Nauk SSSR **120** (1958), 960–962. (Russian)

Let $d(m)$ denote the number of divisors of m, and let $r(m)$ denote the number of representations of m by $Q(x, y)$, a positive primitive binary quadratic form of discriminant $d<0$. Let

$$f(n) = \sum_{\substack{m=1\\ m\equiv l \,(\mathrm{mod}\, D)}}^{n} d(m), \qquad g(n) = \sum_{\substack{m=1\\ m\equiv n \,(\mathrm{mod}\, D)}}^{n} r(m),$$

where $1\leqq l\leqq D$ and $(l, D)=1$. The author first states that asymptotic formulae for $f(n)$, $g(n)$ as $n\to\infty$, valid for $D<n^{1-\epsilon}$, can be proved by using André Weil's estimate for Kloosterman sums. These are of the form

$$f(n) = F(n, D)+O(n^{1-\alpha}), \qquad g(n) = G(n, D)+O(n^{1-\alpha}),$$

where explicit values for $F(n, D)$, $G(n, D)$ are given, the latter being somewhat complicated. The author goes on to announce results on the mean square deviation of $f(n)$ and $g(n)$ when D varies in an interval less restricted than that mentioned above. Theorem 1. Suppose $n^{\frac{1}{2}}\leqq D_1<n^{1-\eta}$, $D_2=D_1^{1-\zeta}$, $l\leqq n$, where η, ζ are fixed small positive numbers. Then

$$\sum_{\substack{D=D_1\\ (D,l)=1}}^{D_2} (f(n)-F(n, D))^2 = O(n^{2-\tau}D_2D_1^{-2}),$$

for some $\tau>0$, with a similar result for $g(n)$ but without the condition $(D, l)=1$. Several applications are also announced. One is to an asymptotic formula for

$$\sum_{m=1}^{n} d(m)d_k(m+l),$$

where $d_k(\nu)$ denotes the number of expressions for ν as a product of k factors, and $k\geqq 3$. Another is that if D is given and $n>n_0(D)$ then n is representable as $\Pi_1+\Pi_2$, where Π_1 is a product of primes $\equiv 1 \pmod 4$ and Π_2 is a product of primes $\equiv 1 \pmod D$. {Obviously n must be understood to be even if D is even.}

H. Davenport (Cambridge, England)

35:

Yüh, Ming-i. A divisor problem. Sci. Record (N.S.) **2** (1958), 326–328.

It is a question of the lower bound α_3 of α for which $\sum_{n\leq x} d_3(n)=xP_3(\log x)+O(x^\alpha)$, where P_3 is a quadratic polynomial. The author announces the result $\alpha_3\leqq 14/29$, improving on the reviewer's 37/75 [Quart. J. Math. Oxford Ser. **12** (1941), 193–200; MR **3**, 269] and also on the (not cited) value $0.493\cdots$ of R. A. Rankin [ibid. (2) **6** (1955), 147–153; MR **17**, 240]. The paper reproduces the breaking up of the main sum into subsums, but not the actual estimation, said to depend on methods of both van der Corput and Vinogradov. *F. V. Atkinson* (Canberra)

36:

Olkin, Ingram. A class of integral identities with matrix argument. Duke Math. J. **26** (1959), 207–213.

In the same J. **23** (1956), 571–577 [MR **18**, 468, 1118], the reviewer gave an extension of the Ingham-Siegel matrix generalization of the Euler gamma function. In this paper, a new proof is given, based upon elementary matrix transformations, together with some further extensions of interest in multivariate analysis.

R. Bellman (Santa Monica, Calif.)

37:

Erdös, P.; and Lorentz, G. G. On the probability that n and $g(n)$ are relatively prime. Acta Arith. **5** (1958), 35–44 (1959).

The main result is as follows. Let $f(x)$ be a positive increasing function with a piecewise continuous derivative and the following properties: (1) For every positive integer d, the numbers $f(dn)/d$ $(n=1, 2, \cdots)$ are equally distributed mod 1; (2) $f(x)=o(x/\log\log x)$ and $xf'(x)/\log\log\log f(x)\to\infty$ as $x\to\infty$; (3) $f'(y)/f'(x)$ is bounded for $y\geqq x>0$. Let $Q(x)$ be the number of those integers n for which $0<n\leqq x$ and $(n, [f(n)])=1$. Then

$$\lim_{x\to\infty}\frac{Q(x)}{x}=\frac{6}{\pi^2}.$$

T. Estermann (London)

38:

Samko, G. P. Construction of a basis of a field defined by two cubic radicals of Gauss integers. Rostov. Gos. Ped. Inst. Uč. Zap. **4** (1957), 3–23. (Russian)

Let k be the Gaussian field, d_1 a Gaussian integer. In the first sections of the paper a basis of algebraic integers $\in k(d_1^{\frac{1}{3}})=K_1$ with respect to k is constructed. Three subcases (corresponding to the decomposition of the ideal (3) into prime ideals) are considered.

Let further d_2 be another Gaussian integer such that $d_2^{\frac{1}{3}}$ is not contained in K_1. The field $K=k(d_1^{\frac{1}{3}}, d_2^{\frac{1}{3}})$ and its algebraic integers are considered. Five possible subcases are considered and in each of them the factorization of the principal ideal (3) into prime ideals is given. In the last section of the paper a basis of algebraic integers of K with respect to k is given for the first of the five subcases.

{Reviewer's note: The author intends, perhaps in another paper, to discuss also the remaining four subcases, but there is no mention about this intention in the paper reviewed.}

Št. Schwarz (Bratislava)

39:

Kurbatov, V. A. Equations of prime degree. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 3 (26), 50–67. (Romanian)

Translation of a Russian paper [Mat. St. (N.S.) **43 (85)** (1957), 349–366; MR **20** #1665].

40:

Vinogradov, A.; Delone, B.; and Fuks, D. Rational approximations to irrational numbers with bounded partial quotients. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 862–865. (Russian)

For an indefinite quadratic form $f(x, y)$ with determinant 1 put $m(f)=\inf|f(x, y)|$, where x, y run through all integer values not both zero. For real numbers θ put

$$\mu(\theta) = \lim\inf|y\theta-x|,$$

where x, y are integers and $y\to\infty$. The authors show that the set of values taken by $\mu(\theta)$ as θ varies (the spectrum of μ) is a subset of the set of values taken by $m(f)$ as f varies (the spectrum of m). This is, of course, an immediate consequence of Mahler's work [Proc. Roy. Soc. London Ser. A **187** (1946), 151–187; Nederl. Akad. Wetensch. Math. Proc. **49** (1946), 331–343, 444–454, 524–532, 622–631; MR **8**, 195, 12] on bodies with automorphs, but the authors present a proof more or less from first principles but along the obvious lines. It then appears that the authors intended to show that the two spectra are in fact identical, but clearly a fair-sized chunk of manuscript got lost between the bottom of page 864 and the top of page 865, and it is not clear to the reviewer how this is to be replaced. Finally, they deduce from a result of Marshall Hall, Jr. [Ann. of Math. (2) **48** (1947), 966–993; MR **9**, 226] that the spectrum for m contains a complete interval to the right of the origin.

{The reviewer takes the opportunity of remarking that he was told by Marshall Hall of this application of his result some years ago, and, indeed, this application provided the motivation for Hall's paper.}

J. W. S. Cassels (Cambridge, England)

41:

Groemer, Helmut. Eine Bemerkung über Gitterpunkte in ebenen konvexen Bereichen. Arch. Math. **10** (1959), 62–63.

Theorem: Let S be a closed 2-dimensional convex set symmetric about the origin. Suppose that the boundary of S is a curve with continuously varying curvature and affine-length λ. Then S contains a point with integral co-ordinates other than the origin provided that $\lambda^3\geqq 288$. The constant 288 is the best possible. The proof depends on the lower bound $\lambda^3/72$ for the area of any circumscribed hexagon [Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer, Berlin–Göttingen–Heidelberg, 1953; MR **15**, 248] and the characterization of the lattice-constant of S in terms of circumscribed hexagons [Reinhardt, Abh. Math. Sem. Hansische Univ. **10** (1934), 216–230].

J. W. S. Cassels (Cambridge, England)

42:

Bombieri, Enrico. Sull'approssimazione di numeri algebrici mediante numeri algebrici. Boll. Un. Mat. Ital. (3) **13** (1958), 351–354. (English summary)

Elementary deduction of a lower bound for $|\xi_1-\xi_2|$, where ξ_1, ξ_2 are nonconjugate algebraic numbers. A less precise result was given by A. Brauer [Jber. Deutsch. Math Verein **38** (1929), 47; for a more complete presentation, see J. Reine Angew. Math. **160** (1929), 70–99, especially pp. 75–78].

C. G. Lekkerkerker (Amsterdam)

ALGEBRAIC GEOMETRY

See also 98, 229, 310.

43:

Chisini, Oscar. La superficie cubica. II. Period. Mat. (4) **35** (1957), 286–300.

This paper is a continuation of a previous article by the author [same Period. **35** (1957), 202–218; MR **20** #864] and is dedicated to monoidal and ruled cubic surfaces, whose

principal properties he gives. The last part of the article is devoted to Sylvester's canonical form of the equation of a general cubic surface. *M. Piazzolla-Beloch* (Ferrara)

44:

Galafassi, Vittorio Emanuele. La superficie cubica generale reale. Period. Mat. (4) **36** (1958), 1–18.

This paper is connected with two articles of O. Chisini [same period (4) **35** (1957), 202–218, 286–300; MR **20** #864] summarizing the properties of cubic surfaces in the complex domain, and deals with questions of reality of non-singular cubic surfaces in a clear and suggestive manner. Starting from the consideration of the 27 lines of the general cubic surface, the author gives an account of the five different types of real non-singular cubic surfaces divided in two groups (the first of four types, the second of one) and gives the plane representation of the surfaces of the first group, from which follow the real properties of the surfaces of the types it contains. He then examines separately the remaining type (which constitutes the second group). The final section is devoted to Geiser's stereographic projection of the general cubic surface, with Zeuthen's application to the real domain.

M. Piazzolla-Beloch (Ferrara)

45:

Facciotti, Guido. Sulle quartiche sghembe di seconda specie. Period. Mat. (4) **36** (1958), 93–109.

Nel presente articolo sono richiamate alcune delle principali proprietà delle quartiche sghembe di seconda specie, dandone la rappresentazione parametrica. Quindi si passa a considerare le corde principali e il centro di Bertini e poi si stabilisce il teorema di Study. Si studia infine la proiezione della curva da un suo punto, da cui si trae un'applicazione alla determinazione del punto doppio di una cubica piana razionale, data mediante speciale rappresentazione parametrica.

M. Piazzolla-Beloch (Ferrara)

46:

Porcu, Livio. Il metodo di "piccola variazione" in problemi concernenti le curve algebriche piane reali. I. Period. Mat. (4) **36** (1958), 156–174.

Esposizione di carattere didattico del metodo di "piccola variazione" con considerazioni preliminari sulle curve algebriche piane reali.

M. Piazzolla-Beloch (Ferrara)

47:

Lavis, A. Sur des involutions cycliques appartenant à certaines surfaces algébriques. I. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 708–722.

L'Autore considera una particolare superficie algebrica F di S_{r+4} trasformata in sè da un'omografia ciclica d'ordine $2r+1$ che induce su F un'involuzione I_{2r+1} dotata di un numero finito di punti uniti; e costruisce una superficie F' immagine di I_{2r+1} sulla quale ai punti uniti di I_{2r+1} corrispondono punti doppi biplanari, dei quali determina la struttura. Studia inoltre i sistemi di curve tracciate su F'. *D. Gallarati* (Genova)

48:

Lavis, A. Sur des involutions cycliques appartenant à certaines surfaces algébriques. II. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 767–779.

L'Autore considera una particolare superficie algebrica F di S_{r+4} trasformata in sè da un'omografia ciclica d'ordine $2r+1$ che induce su F un'involuzione I_{2r+1} che possiede un numero finito di punti uniti; e costruisce una superficie F' immagine di I_{2r+1} sulla quale i punti di diramazione sono punti $(r+1)$-pli con cono tangente composto di un cono razionale d'ordine r e di un piano che contiene una generatrice del cono.

D. Gallarati (Genova)

49:

Gaeta, Federico. Sull'equazione canonica di un complesso C^g_{n-d-1} di sottospazi S_d di S_n. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 389–394.

È noto che ogni ciclo puro di dimensione massima della Grassmanniana $G(d, n)$ degli S_d di $S_n(K)$ (K corpo base commutativo di caratteristica zero), ciclo immagine di un complesso d'ordine g di sottospazi S_d di S_n, è la completa intersezione di $G(d, n)$ con una g-ica dello spazio ambiente; ma questa g-ica, non appena $g>1$ (e $0<d<n-1$), non è unica. L'Autore dimostra l'esistenza d'una g-ica canonica secante, individuata intrinsecamente dal complesso e covariante per il gruppo delle omografie non degeneri che mutano in sè la Grassmanniana. *D. Gallarati* (Genoa)

50:

Igusa, Jun-ichi. Abstract vanishing cycle theory. Proc. Japan Acad. **34** (1958), 589–593.

The author's previous abstract vanishing cycle theory [Amer. J. Math. **78** (1956), 745–760; MR **18**, 936] is here simplified and generalized to the unequal characteristic case. A nonsingular curve is defined over the quotient field of a complete discrete valuation ring, the specialized curve is assumed absolutely irreducible and nonsingular except for one ordinary double point, and the author considers the relations between the points of a given order (prime to the characteristic of the residue field) on the jacobian variety of the original curve and their specializations on the generalized jacobian of the specialized curve.

M. Rosenlicht (Evanston, Ill.)

51:

Mallol, Rafael. On the behavior of an algebraic variety in the extensions of the reference field. Collect. Math. **10** (1958), 125–136. (Spanish)

A re-exposition, ab initio, of how a k-variety decomposes into irreducible parts when the field k is extended.

M. Rosenlicht (Evanston, Ill.)

52:

Igusa, Jun-ichi. A remark on the theory of the base. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 139–143.

The author proves the following theorems. (1) If A and B are isogenous abelian varieties such that the semi-group of positive divisor classes modulo numerical equivalence of one of them has a finite set of generators, then so has the semi-group of the other. (2) If an abelian variety A^n contains at least one simple abelian subvariety whose Picard number is $\geqq 2$, then the semi-group of A has no finite set of generators. An example is given of such an A. (3) Let V^n be a normal projective variety; then the semi-group of positive divisor classes on V is finitely generated if and only if there is a positive integer m such that every subvariety C^{n-1} of degree $>m$ is numerically equivalent to a reducible variety.

F. D. Quigley (New Orleans, La.)

LINEAR ALGEBRA

See also 69, 73, 82. 284, 316, 424, 426.

53:

Friedman, Bernard. *n*-commutative matrices. Math. Ann. **136** (1958), 343–347.

The author considers $m\times m$ complex matrices M. Such a matrix is "broken-diagonal" in case its i,j-entry is zero except possibly for $j=i+1$ and for $i=1$, $j=m$. A set of matrices $M_1,\cdots,M_p$ is "n-commutative" ($n\geqq 1$) in case the matrices $M_{i_1}\cdots M_{i_n}$ are commutative. Hypothesis: $M_1,\cdots,M_p$ are k-commutative, but not n-commutative for $n<k$; $M_1,\cdots,M_p$ are not simultaneously reducible; some linear combination of them is non-singular. Conclusion: They are $k\times k$ and can be simultaneously transformed into broken-diagonal form. *C. Davis* (Providence, R.I.)

54:

Marcus, Marvin. All linear operators leaving the unitary group invariant. Duke Math. J. **26** (1959), 155–163.

Let M_n be the linear space of all $n\times n$ matrices over the field of complex numbers, and denote by O_n the unitary group in M_n. The present paper is concerned with the structure of the set Ω_n of linear transformations on M_n to M_n which map O_n into itself. It is shown that Ω_n is a group and that $T\in\Omega_n$ if and only if there exist unitary matrices U, V such that, for every $A\in M_n$, we have $T(A)=UAV$ or $T(A)=UA'V$. *L. Mirsky* (Sheffield)

55:

Khan, N. A. A note on *S*-matrices. Gaṇita **8** (1957), 17–21.

An $n\times n$ S-matrix A corresponding to the vector $u=(u_1,\cdots,u_n)$ with $\sum u_i\neq 0$ has the property that the vector $uA=uA'$ and has equal components [see Weiner, Bull. Amer. Math. Soc. **61** (1955), 314]. It is shown that these matrices form a group if the product of two such matrices A, B is defined as AUB where U is the diagonal $(u_1,\cdots,u_n)$. In contradiction to a statement by the author, this group can have elements of finite order, e.g. when A is a permutation matrix and $u_i=1$.
O. Taussky-Todd (Pasadena, Calif.)

56:

Saxena, Bhagwat Swarup. Symbolic matrix integration. Gaṇita **8** (1957), 105–124.

Let $X=(x_{ij})$ be an $m\times n$ matrix and let y be a function of the variables x_{ij}. Then $\int y dx$ is defined as the $m\times n$ matrix whose (i,j)th element is $\int y dx_{ij}$. Further, if $Y=(y_{ij})$ is an $m\times n$ matrix and the y_{ij} are functions of a single variable t, then $\int Y dt$ is defined as the $m\times n$ matrix whose (i,j)th element is $\int y_{ij}dt$. The author develops a series of elaborate identities involving integrals of the above two types, but the details of the discussion are not altogether clear to the reviewer. *L. Mirsky* (Sheffield)

57:

Drboglav, Karel [Drbohlav, Karel]. Über das Minimum einer gewissen Linearform. Czechoslovak Math. J. **8** (**83**) (1958), 190–196. (Russian. German summary)

Notational definitions: Large Latin letters represent $m\times n$ matrices; $A\geqq 0$ if all of its elements are non-negative; $U\sim V$ if $\sum_{i=1}^m u_{ij}=\sum_{i=1}^m v_{ij}$ for each j and $\sum_{j=1}^n u_{ij}=\sum_{j=1}^n v_{ij}$ for each i, $(U,V)\equiv\sum_{i=1}^m\sum_{j=1}^n u_{ij}v_{ij}$.

Problem: Given matrices K and $B\geqq 0$, find A in the class of all matrices $X\geqq 0$, and $X\sim B$, such that (K,A) is a minimum. In solving this problem the author uses the partial ordering $U\prec V$ if $v_{ij}\geqq 0$ implies $v_{ij}\geqq u_{ij}\geqq 0$ and $v_{ij}\leqq 0$ implies $v_{ij}\leqq u_{ij}\leqq 0$. The author generates a sequence of matrices which will tend to A, and if the elements of B are rational, then A is obtained in a finite number of steps.
A. W. Goodman (Lexington, Ky.)

58:

Fiedler, Miroslav. On some properties of Hermitian matrices. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 168–176. (Czech. Russian and English summaries)

A square matrix A (with complex elements) is called decomposable if by permutations of the rows and columns it can be transformed to the form $\begin{pmatrix}A_1 & 0\\ B & A_2\end{pmatrix}$, A_1,A_2 being square matrices of order at least one. Otherwise it is called indecomposable. We shall say that the order of decomposability $\rho(A)$ of A is $r-1$ if by row and column interchanges A can be transformed into the form

$$\begin{pmatrix}A_{11} & 0 & \cdots & 0\\ A_{21} & A_{22} & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ A_{r1} & A_{r2} & \cdots & A_{rr}\end{pmatrix}$$

where the square matrices A_{11}, A_{22}, $\cdots$, A_{rr} of order at least one are indecomposable.

If $A=(a_{ij})$ is a complex matrix we define $A_-=(u_{ij})$, where $u_{ij}=\operatorname{Re}a_{ij}$ if $\operatorname{Re}a_{ij}<0$, $u_{ij}=0$ if $\operatorname{Re}a_{ij}\geqq 0$. If $A=(a_{ij})$, $B=(b_{ij})$ are two matrices of the same order we define $A\circ B=(a_{ij}b_{ij})$. The following theorems are proved.

Suppose that A is Hermitian and the least eigenvalue has the multiplicity s. Suppose further that to this eigenvalue there corresponds at least one positive eigenvector. Then $\rho(A_-)\leqq s-1$.

Let A be a positive definite Hermitian matrix with $\rho(A)=r$. Then the matrix $H=A\circ A'^{-1}$ has the least eigenvalue equal to 1. This eigenvalue is of the multiplicity $r+1$ and the corresponding eigenvector is j, where $j'=(1,1,\cdots,1)$.

If A is a positive definite Hermitian matrix and H is as above, then $\rho(H_-)=\rho(A)$.

In the case that A is a real symmetric matrix these results give relations between the signs of the elements of A, A^{-1} and those of the coordinates of the eigenvectors corresponding to the least eigenvalue.
Št. Schwarz (Bratislava)

59:

Kisyński, J. Sur la méthode des approximations successives pour un système de *n* équations à *n* inconnues. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 683–687.

Let E_i be a complete metric space with a metric d_i, E the Cartesian product $E=E_1\times\cdots\times E_n$, $L_{ik}\geqq 0$ constants and $F=F(L_{11},\cdots,L_{nn})$ the family of maps $f:x=(x_1,\cdots,x_n)\to y=(y_1,\cdots,y_n)$ of E into itself satisfying the Lipschitz condition $d_i(y_i',y_i'')\leqq\sum_{k=1}^n L_{ik}d_k(x_k',x_k'')$ for all pairs of points x', x'' and their images y', y''. Necessary and sufficient for every map f of F to have a fixed point x^0, $f(x^0)=x^0$, is that the principal minors of the matrix $(\delta_{ik}-L_{ik})$ be positive; in which case x^0 is unique and is

the limit of the successive approximations $x^1=f(x)$, $x^2=f(x^1)$, $\cdots$. The proof is based on a lemma asserting that if $a_{ik}\leq 0$ for $i\neq k$, then the principal minors of the matrix (a_{ik}) are positive if and only if there exist positive numbers $\alpha_1, \cdots, \alpha_n$ satisfying $\sum_{k=1}^n a_{ik}\alpha_k>0$ for $i=1, \cdots, n$. *P. Hartman* (Baltimore, Md.)

60:

Haimovici, Corina. Matrizen zweiter Ordnung mit Elementen aus dem Körper der Restklassen mod p. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 11–19. (Romanian and Russian summaries)

(1) Number theoretic criterion that A have a root in the coefficient field K_p. (2) Study of $X^k\equiv A$. (3) Proof that $A^p\equiv A$, if roots of A lie in K_p.
J. L. Brenner (Palo Alto, Calif.)

61:

Paasche, Ivan. Verallgemeinerung zweier Sätze von Castaldo. Arch. Math. **9** (1958), 410–411.

Let D_ν be the diagonal matrix diag $(A_{0\nu}, \cdots, A_{n\nu})$, and let C be the matrix $a_0E_{12}+a_1E_{23}+\cdots+a_{n-1}E_{n-1,n}+a_nE_{n,1}$. Here, E_{ij} are the $n+1\times n+1$ matrix units. Then the matrix $A=\sum D_\nu C^\nu$ is essentially similar to a matrix the elements of which do not involve the a_i, but only their product a.

The author points out the corollary that det A is a function of the A_{ij} and a only.
J. L. Brenner (Menlo Park, Calif.)

62:

Castaldo, Domenico. Matrici cicliche e relativi determinanti. Ricerca, Napoli (2) **7** (1956), 29–40.

The results of this paper are contained in the paper of Paasche reviewed above.
J. L. Brenner (Menlo Park, Calif.)

63:

Reiman, I. Über ein Problem von K. Zarankiewicz. Acta. Math. Acad. Sci. Hungar. **9** (1958), 269–273.

K. Zarankiewicz [Colloq. Math. **2** (1951), 301; problème 101], avait posé la question suivante: "Soit A_n une matrice carrée de n lignes dont tous les éléments sont des 0 et des 1. Quel est le plus petit nombre de 1 contenus dans A_n pour lequel il existe nécessairement un mineur du second ordre M_2 uniquement composé de 1?" L'auteur perfectionne les résultats déjà obtenus par S. Hartman, J. Mycielski et C. Ryll-Nardzewski [ibid. **3** (1954), 84–85] et par T. Kövári, Vera T. Sós et P. Turán [ibid. 50–57; MR **16**, 456]. Une matrice est "saturée" si (i) elle ne contient aucun mineur M_2, (ii) par la substitution d'un 1 à un 0, la nouvelle matrice contient au moins un M_2. Pour construire une matrice "saturée" il suffit de dresser la table qui définit, suivant le mode habituel, les incidences d'un espace projectif fini. Soit $k_2(n_1, n_2)$ le nombre minimum de 1 que doit contenir une matrice de n_1 lignes et de n_2 colonnes pour que celle-ci possède au moins un mineur M_2. Utilisant la géométrie sur GF(p^m) et le nombre, s, des 1 dans la matrice "saturée" correspondante, l'auteur démontre que $k_2(n_1, n_2)=s+1$, où s est la racine positive de l'équation $(s-n_1)s=n_1n_2(n_2-1)$. Si la matrice devient carrée cette borne supérieure est asymptotique à celle de Paul Turán. Exemple pour $n=21$. *A. Sade* (Marseille)

64:

Taussky, Olga. A note on the group commutator of A and A^*. J. Washington Acad. Sci. **48** (1958), 305.

A short proof that matrix A commutes with A^* (its conjugate transpose) if A commutes with AA^*-A^*A.
D. C. Kleinecke (Livermore, Calif.)

65:

Thompson, R. C. On matrix commutators. J. Washington Acad. Sci. **48** (1958), 306–307.

If hermitian matrix M has trace zero then

$$M = AA^*-A^*A$$

for some matrix A and its conjugate transpose A^*. Matrix A can be assumed to be nonsingular or of the form $XY-YX$. *D. C. Kleinecke* (Livermore, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 284.

66:

Jacobson, N. Composition algebras and their automorphisms. Rend. Circ. Mat. Palermo (2) **7** (1958), 55–80.

The problem of finding the quadratic forms permitting composition (Hurwitz's problem) and finding the algebras $\mathfrak{C}$ arising from such forms (these are called composition algebras) is solved completely even though it has been solved elsewhere [see Kaplansky, Proc. Amer. Math. Soc. **4** (1953), 956–960; MR **15**, 596] because the analysis of the composition algebras given here is essential to the study of the automorphisms. The study of the automorphisms is the main purpose of the paper. Following is a summary of what appear to be the main results obtained.

A "split composition algebra" is one which contains zero divisors, and it is proved that any two split composition algebras of the same dimension are isomorphic. The split algebras are then determined. An automorphism τ is called a reflection if $\tau^2=1$ and $\tau\neq 1$. A reflection is called split if the subalgebra $\mathfrak{B}$ of elements fixed by τ is split. If $\mathfrak{C}$ is a quaternion or Cayley algebra then every automorphism of $\mathfrak{C}$ is a product of reflections which may be taken to be split if $\mathfrak{C}$ is split Cayley; every automorphism is inner and is a rotation in $\mathfrak{C}$ (relative to N). If G is the automorphism group of $\mathfrak{C}$ and $\mathfrak{B}$ a subalgebra of $\mathfrak{C}$ then $G_{\mathfrak{B}}$, the Galois group of $\mathfrak{C}$ over $\mathfrak{B}$ is determined for several choices of $\mathfrak{C}$ and $\mathfrak{B}$. When $\mathfrak{C}$ is split Cayley, G is a simple group. For $\mathfrak{C}$ any composition algebra, the only invariant subspaces of $\mathfrak{C}$ relative to G are $\mathfrak{C}$, 0, $\Phi 1$ and $\mathfrak{C}_0$, where Φ is the base field of $\mathfrak{C}$ and $\mathfrak{C}_0$ is the subspace of elements orthogonal to 1. Special properties are obtained by specializing Φ. The following question is raised: Is the group of automorphisms of any Cayley division algebra over an algebraic number field Φ simple? It is known that this is not true over an arbitrary field. The answer is yes if Φ is the field of real numbers. To prove this the author proves a more general result which may be useful in answering the above question. *L. A. Kokoris* (Chicago, Ill.)

67:

Harada, Manabu. Some remarks on E-sequences in Noetherian rings. J. Inst. Polytech. Osaka City Univ. Ser. A **9** (1958), 39–41.

Let A be a commutative noetherian ring, $\mathfrak{b}$ an ideal in A,

E a finite A-module with $bE \neq E$. A sequence of elements $a_1, a_2, \cdots, a_k$ is called a b-E sequence if $b_iE: b_{i+1} = b_iE$, $i = 0, \cdots, k-1$, where $b_i = (b_1, \cdots, b_i)$. The author proves that every b-E sequence can be extended to a maximal one and that all maximal b-E sequences have the same length. This result essentially has already appeared [Rees, Proc. Cambridge Philos. Soc. **52** (1956), 605–610; MR **18**, 277] but the author gives a somewhat different proof.

A. Rosenberg (Evanston, Ill.)

68:

Borevič, Z. I. On the proof of the principal ideal theorem. Vestnik Leningrad. Univ. **12** (1957), no. 13, 5–8. (Russian. English summary)

Using a result from his earlier paper (Dokl. Akad. Nauk SSSR **91** (1953), 193–195; MR **15**, 598], the author offers a new proof of the principal ideal theorem [Abh. Math. Sem. Hamburg. Univ. **7** (1930), 14–36; **10** (1934), 349–357].

R. A. Good (College Park, Md.)

69:

Pham, Daniel. Sur les anneaux indexables. Ann. Sci. École Norm. Sup. (3) **75** (1958), 81–105.

Let A be a ring with unit. The class (x) for an element x in A consists of all the elements of the form pxq with p, q units of A. The author calls A indexable if its classes may be totally ordered in such a way that $(0) \leq (x)$ if $x \neq 0$; $(xy) \leq (x)$ and $(xy) \leq (y)$; and there is only a finite number of classes between (0) and any class $(x) \neq (1)$. If this ordering is unique A is said to be strictly indexable. A simple ring with minimum condition is strictly indexable with (x) simply consisting of all the elements of a given rank. The author then studies direct sums, ideal direct summands, and homomorphic images of indexable rings and shows that these are again indexable (the first only under the hypothesis that there are only finitely many classes in each summand). The last part of the paper is devoted to the problem of decomposing an indexable ring into a direct sum of rings with a more transparent structure. The results are too complicated to be reproduced here. The main tool in this part is the notion of a "kernel": a two sided ideal N generated by a class which is minimal in some ordering. For such ideals, $N^2 = N$ or 0.

A. Rosenberg (Evanston, Ill.)

70:

Wall, Drury W. Characterizations of generalized uni-serial algebras. Trans. Amer. Math. Soc. **90** (1959), 161–170.

Let A be a finite-dimensional algebra with unit element over a field. A primitive left [or right] ideal is called dominant if it is dual to some primitive right [or left] ideal, that is, if it is injective. A is called QF–2, QF–3*, or QF–3 if every primitive (left or right) ideal is embeddable into some dominant primitive ideal, a direct sum of copies of some dominant primitive ideal, or a direct sum of copies of dominant primitive ideals, respectively [Thrall, Trans. Amer. Math. Soc. **64** (1948), 173–183; MR **10**, 98], while A is quasi-Frobenius if and only if every primitive ideal is dominant, and A is Frobenius if and only if the left A-module A and the right A-module A are dual to each other. A is said to be generalized uni-serial if every primitive ideal has only one composition series; if moreover A is primary decomposable then we call A uni-serial. The well-known characterization of uni-serial algebras is: A is uni-serial if and only if, for every two-sided ideal Z of A, the residue class algebra A/Z is Frobenius [Nakayama, Proc. Imp. Acad. Tokyo **16** (1940), 285–289; MR **2**, 245]. The author gives the following similar characterization of generalized uni-serial as well as uni-serial algebras: (1) A is generalized uni-serial if and only if every residue class algebra A/Z of A is either QF–2 or QF–3*. (2) A is uni-serial if and only if every residue class algebra of A is quasi-Frobenius. It is to be added that the latter theorem has been obtained not only by Osima but also by Ikeda [Osaka Math. J. **3** (1951), 227–239; MR **13**, 719].

G. Azumaya (Evanston, Ill.)

71:

Wall, Drury W. Characterizations of generalized uni-serial algebras. II. Proc. Amer. Math. Soc. **9** (1958), 915–919.

Extending his earlier theorem on generalized uni-serial algebras [reviewed above], the author shows that A is generalized uni-serial if and only if every residue class algebra of A is QF–3. The same result has also been obtained independently by Morita [Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142; MR **20** #3183; Theorem 17.8].

G. Azumaya (Evanston, Ill.)

72:

Kertész, A. Eine Charakterisierung der halbeinfachen Ringe (Ergänzung zu meiner Arbeit "Beiträge zur Theorie der Operatormoduln"). Acta Math. Acad. Sci. Hungar. **9** (1958), 343–344.

A ring is semisimple if and only if it has a right unit element and the left annihilator of each nonzero element is the intersection of a finite number of maximal left ideals.

Graham Higman (Oxford)

73:

Andreoli, Giulio. Commutazione ed anticommutazione. Numeri di Dirac e di Hafner. Ricerca, Napoli (2) **8** (1957), Gennaio-Giugno, 3–24.

Die vorliegende Arbeit enthält einige Bemerkungen über diracsche und hafnersche Zahlen im Zusammenhang mit endlichen und unendlichen Matrizen und anschliessend Überlegungen über abstrakte Algebren. Eine Algebra A heisst bekanntlich kommutativ [antikommutativ] wenn $a, b \in A \Rightarrow ab - ba = 0$ [$a, b \in A \Rightarrow ab + ba = 0$]. Ferner wird A als eine diracsche [hafnersche] Algebra im bezug auf eine Zahl k ($\neq 0$) bezeichnet, wenn für geeignete Elemente $a, b \in A$ die Beziehung $ab - ba = k\varepsilon$ [$ab + ba = k\varepsilon$] besteht, wobei ε den Modul von A bedeutet. Im Wesentlichen handelt es sich um Erweiterungen dieser Begriffe auf den Fall der Verknüpfungsregeln $ab = \lambda ba$, $ab = b^\rho a^\sigma$ mit numerischen Werten λ; ρ, σ und $ab = \Theta ba$, $\Omega(ab) = \Omega b \Omega a$ mit Operatoren Θ, Ω. Es wird gezeigt, dass die angeführten mit λ; ρ, σ gebildeten Verknüpfungsregeln keine Erweiterungen der kommutativen und antikommutativen Algebren ergeben. Dagegen führen die mit den Operatoren Θ, Ω definierten Multiplikationen auf allgemeinere Begriffe.

O. Borůvka (Brno)

74:

Nakayama, Tadasi. A remark on the commutativity of algebraic rings. Nagoya Math. J. **14** (1959), 39–44.

Call a ring F an N-ring (reviewer's terminology) if it is commutative with unit element, contains a (possibly zero)

subring K which is an algebraically closed field modulo every prime ideal of K, and contains a finite (possibly empty) set of elements which, together with K, generate F. The author's principal result is that an algebra over an N-ring F is necessarily commutative whenever (*) there is a function $a \to p_a(\lambda)$ from R into the polynomial ring $F[\lambda]$ such that $a-a^2p_a(a)$ is central for every $a \in R$. This generalizes results of Herstein [Amer. J. Math. **75** (1953), 864–871; MR **15**, 392], who took F to be the integers, and of the author [Abh. Math. Sem. Univ. Hamburg **20** (1955), 20–27; MR **17**, 341], but still leaves open the interesting problem of how to characterize intrinsically the class of all F for which (*) implies the commutativity of R.

In the special case when R is an algebraic ring over F in the sense of the reviewer [Canad. J. Math. **8** (1956), 341–354; MR **17**, 1179], i.e. when every element of R satisfies a polynomial equation over F with unity as lowest nonzero coefficient, a lemma of the reviewer [loc. cit., Theorem 6.1] asserts that the hypothesis (*) is equivalent to the centrality of all nilpotent elements of R. Thus, as the author points out, his theorem implies the following generalization of earlier results of Arens and Kaplansky [Trans. Amer. Math. Soc. **63** (1948), 457–481; MR **10**, 7], Herstein [Proc. Amer. Math. Soc. **5** (1954), 620; MR **16**, 5], Jacobson [Ann. of Math. (2) **46** (1945), 695–707; MR **7**, 238] and the reviewer [loc. cit.]: if R is an algebraic ring over an N-ring and has all its nilpotent elements central, then R is commutative.

M. P. Drazin (Baltimore, Md.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

75:

Sunouchi, Haruo. Infinite Lie rings. Tôhoku Math. J. (2) **8** (1956), 291–307.

This paper presents a new proof of the result of R. V. Kadison which specifies all the closed normal subgroups of the unitary group of factors. The present discussion is based on the relationship between normal subgroups and Lie ideals, which is essentially that if I is a normal subgroup of the unitary group U then the Lie ring of I is a Lie ideal in the Lie ring of U.

The Lie ideals in a factor are determined by considering the various derived Lie rings and then the question as to the existence and description of the corresponding normal subgroup is answered. *F. J. Murray* (New York, N.Y.)

76:

Patterson, E. M. On certain types of derivations. Proc. Cambridge Philos. Soc. **54** (1958), 338–345.

In this short paper the author provides some answers to the question: what types of non-associative rings R admit non-singular algebraic derivations. His principal results are: (i) if R is of characteristic 0 and D a derivation with $D^k=I$, then k is a multiple of 6; (ii) with some exceptions for characteristics which are small primes, if R admits a derivation of degree ≤ 3, then R is, in a sense, nilpotent. On the other hand, there exist simple algebras of prime characteristic with many non-singular derivations whose orders are not multiples of 6, and there exist simple algebras with derivations of degree 4.

W. G. Lister (Oyster Bay, N.Y.)

HOMOLOGICAL ALGEBRA

77:

Norguet, François. Sur l'homologie associée à une famille de dérivations. C. R. Acad. Sci. Paris **247** (1958), 1081–1083.

The author states a sufficient condition for the following proposition to hold for all m and all $\{\omega_i\}$: $(\wedge\omega_i)\wedge m=0$ implies $m=\sum \omega_i \wedge m_i$ for some $\{m_i\}$. Here the ω_i are elements of a finitely generated free module A^n over a commutative ring A with unit; m is an element of $M\otimes E$ where M is an A-module and E is the exterior algebra generated by A^n; and $\wedge$ denotes exterior product. If $p=1$ the resulting theorem reduces to a result of de Rham [Comment. Math. Helv. **28** (1954), 346–352; MR **16**, 402]. The problem can be generalized to a problem in abelian categories, but this note makes no attempt to solve the generalized problem. *D. Zelinsky* (Evanston, Ill.)

78:

Amitsur, S. A. Simple algebras and cohomology groups of arbitrary fields. Trans. Amer. Math. Soc. **90** (1959), 73–112.

Let F be a finite algebraic extension field of a field C, let F^n be the n-fold tensor product $F\otimes_C \cdots \otimes_C F$ and let $\varepsilon_i: F^n \to F^{n+1}$ be the additive mapping defined by $\varepsilon_i(a_1\otimes\cdots\otimes a_n)=a_1\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_n$. Let F^{n*} be the group of units in F^n and, for x in F^{n*}, define $\delta_n(x)=\varepsilon_1(x)\varepsilon_2(x)^{-1}\varepsilon_3(x)\varepsilon_4(x)^{-1}\cdots$. Then the groups F^{n*} and the mappings δ_n form a complex whose homology groups $H^n=\ker \delta_{nH}/\mathrm{im}\ \delta_n$ have the following significance. (1) If F is normal, separable over C with Galois group G, then H^n is the nth cohomology group $H^n(G, F^*)$ of the group G. (2) For general F, H^2 is isomorphic to the Brauer group of central simple C-algebras with F as splitting field.

The isomorphism involved in the major result (2) is the composite of a chain of correspondances: algebra split by $F \to$ "semilinear mapping" of F^2 to $F^3 \to$ closed, relatively cyclic bimodule $\to$ unit in F^3. Similar considerations give an analogous formula for the group of algebras simultaneously split by two fields.

Section 7 on the analogue of (1) for purely inseparable fields of exponent one is in error, primarily because the related complexes $\mathscr{C}(F^+)$ and $\mathscr{C}(N^+)$ are, in fact, acyclic, invalidating lemma 7.1 and theorem 7.2. Theorem 7.4 is also incorrect. *D. Zelinsky* (Evanston, Ill.)

79:

Harada, Manabu. The weak dimension of algebras and its applications. J. Inst. Polytech. Osaka City Univ. Ser. A **9** (1958), 47–58.

In the first section of this note, analogues of some of the results of Rosenberg and Zelinsky [Trans. Amer. Math. Soc. **82** (1956), 85–98; MR **17**, 1181] and Eilenberg, Rosenberg and Zelinsky [Nagoya Math. J. **12** (1957), 71–93; MR **20** #5229] for algebra dimension are derived for the weak algebra dimension. The only noteworthy difference is that if Λ is a field of transcendence degree n which can be separably generated, w. dim $\Lambda=n$, whereas dim Λ may be $>n$. Next the author considers algebras all of whose scalar extensions are of weak dimension 0. He shows, among other results, that if such an algebra is either commutative algebraic or an integral domain it is locally separable. Next the author adapts a theorem of Amitsur

[Canad. J. Math. 8 (1956), 355–361; MR 17, 1179] to show that if Λ is a rational function field and R an algebra with no nil ideals, then $\Lambda \otimes R$ is Jacobson semi-simple. From this he manufactures several examples of algebras for which Λ^e is Jacobson semi-simple. Since it was shown earlier that w. dim $\Lambda = 0$ if and only if Λ^e is regular, this last class of algebras is a generalization of those of zero weak dimension. *A. Rosenberg* (Evanston, Ill.)

GROUPS AND GENERALIZATIONS

See also 450.

80:

Boone, William W. The word problem. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 1061–1065.

The author obtains an improvement and simplification of his earlier results and methods [see Indag. Math. **16** (1954), 231–237, 492–497; **17** (1955), 252–256; MR **16**, 564]. A central result gives a recursive map from Thue systems T to finite presentations G of groups, with the property that, for all A, B in a Thue system T, the equality of A and B is equivalent to the equality of the elements represented by words A', B', corresponding recursively to A and B, in the group defined by the presentation G corresponding to T. *R. C. Lyndon* (Ann Arbor, Mich.)

81:

Moser, W. O. J. Abstract definitions for the Mathieu groups M_{11} and M_{12}. Canad. Math. Bull. **2** (1959), 9–13.

The contents of this paper are precisely indicated by its title. *Graham Higman* (Oxford)

82:

Hua, Loo-keng. A subgroup of the orthogonal group with respect to an indefinite quadratic form. Sci. Record (N.S.) **2** (1958), 329–331.

Let F be a field of characteristic $\neq 2$, and

$$\Delta = [1, \cdots, 1, -1, \cdots, -1]$$

a diagonal matrix with l positive 1's and m negative 1's on its diagonal; $n = l+m$. Consider $n \times n$ matrices Γ with elements in F satisfying $\Gamma\Delta\Gamma' = \Delta$, where Γ' denotes the transpose of Γ. Write

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, and D are, respectively, $l \times l$, $l \times m$, $m \times l$, and $m \times m$ matrices. The following theorem is proved: Those Γ with det A equal to the sum of the square elements of F form a (multiplicative) group. Also those with both det A and det D equal to the sum of the square elements form a group. *R. Ree* (Kingston, Ont.)

83:

Sesekin, N. F.; and Širokovskaya, O. S. A class of bigraded groups. Mat. Sb. N.S. **46** (88) (1958), 133–142. (Russian)

A non-abelian group G distinct from its commutant K is called a B-group provided every (proper) normal subgroup is commutative. A B-group is called a B_1-group or a B_2-group according as G/K is a primary cyclic group or is the direct product of two primary cyclic groups. A necessary and sufficient set of conditions that a group G be a B_2-group is that G be a finite p-group, that K be cyclic of prime order p, that G/K be a direct product of two primary cyclic groups, and that the factor-group of G modulo its center be a direct product of two cyclic groups of order p. A necessary and sufficient set of conditions that a group G be a B_1-group is that G/K be a cyclic group with order a power of the prime p and that G have an abelian subgroup of index p containing K. The class of p-primary B_1-groups is characterized. A group which is different from its commutant and all of whose (proper) subgroups are commutative is either itself commutative or is a finite group as studied by Miller and Moreno [Trans. Amer. Math. Soc. 4 (1903), 394–404]. *R. A. Good* (College Park, Md.)

84:

Auslander, Maurice. Remark on automorphisms of groups. Proc. Amer. Math. Soc. **9** (1958), 229–230.

Let G be a group with center C and let α be an automorphism of G such that α^n is an inner automorphism for some integer $n > 0$. Denote by C_α the subgroup of C consisting of all elements of the form $y^{-1}\alpha(y)$ $(y \in C)$. As x runs through all elements in G which induce the same inner automorphism as α^n, the elements $x^{-1}\alpha(x)$ sweep out a coset of C_α in C denoted by $O(\alpha, n)$. The author proves the following theorem. If all the fixed points of α are in the center of G, then $\alpha^{n^2} = 1$. Further, $\alpha^n = 1$ if and only if $O(\alpha, n) = 1$. *D. Buchsbaum* (Providence, R.I.)

85:

Parízek, Bohumír. Note on structure of multiplicative semigroup of residue classes. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 183–185. (Slovak. Russian and English summaries)

Let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_i \geqq 1$, be the factorization of the integer $m > 1$ into different primes and S_m the multiplicative semigroup of residue classes modulo m. A simple proof of the following theorem is given: S_m is a class sum of disjoint groups if and only if $\alpha_1 = \cdots = \alpha_r = 1$. *Št. Schwarz* (Bratislava)

86:

Aĭzenštat, A. Ya. Defining relations in symmetric semigroups. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 121–142. (Russian)

The semigroups treated are the semigroup under composition of all mappings of a finite set into itself, the semigroup of those mappings of a countable set into itself which disturb only a finite number of elements, and the semigroup obtained from this last by adjoining a mapping which permutes all the elements of the set cyclically. In each case, a set of generators for the semigroup is obtained by adding to a set of generators for the group a mapping which maps both elements of some pair onto one of them, and all other elements onto themselves. Defining relations are found for each of the semigroups in terms of these generators. *Graham Higman* (Oxford)

87:

Bredihin, B. M. Free numerical semigroups with power densities. Mat. Sb. N.S. **46** (88) (1958), 143–158. (Russian)

The Selberg approach is extended to yield an elementary proof of an asymptotic law for the distribution of generating

elements in a free numerical semigroup. Let G be a free commutative semigroup with a countable system P of generators. Let N be a homomorphism of G onto a multiplicative semigroup of numbers such that, for a given number x, only finitely many elements α in G have norm $N(\alpha)$ satisfying $N(\alpha)\leq x$. Let $\nu(x)=\sum_{N(\alpha)\leq x, \alpha\in G}1$ and $\pi(x)=\sum_{N(\omega)\leq x, \omega\in P}1$. Given $\theta>0$, if $\lim_{x\to\infty}(\nu(x)/x^\theta)$ exists and is a positive number C, then C is called the density, relative to the θ power, of the semigroup. The asymptotic law may be expressed as follows: if the free semigroup has the property that $\nu(x)=Cx^\theta+O(x^{\theta_1})$, where $C>0$, $\theta>0$, and $\theta_1<\theta$, then

$$\lim_{x\to\infty}\frac{\pi(x)}{x^\theta/\log x}=\theta^{-1}.$$

As special cases there are the classical prime number theorem, theory concerning the distribution of primes among the Gaussian complex numbers, and results on the distribution of zeros of the ζ-function generalized to free semigroups. *R. A. Good* (College Park, Md.)

88:

Aĭzenštat, A. Ya. Defining relations of finite symmetric semigroups. Mat. Sb. N.S. **45** (87) (1958), 261–280. (Russian)

Let Σ_n denote the symmetric semigroup of degree n consisting of all single-valued mappings of the set $I=\{1, 2, \cdots, n\}$ into itself, and let σ_n denote the symmetric group of degree n contained in Σ_n. If $\phi\in\Sigma_n$ and the image set $I\phi$ contains $n-k$ elements, then ϕ is of defect k. N. N. Vorob'ev [Leningrad. Gos. Ped. Inst. Uč. Zap. **89** (1953), 161–166; MR **17**, 943] has shown that the irreducible sets of generators of Σ_n are the sets consisting of a single element of defect one together with an irreducible set of generators of σ_n. Using the usual notation for cycles, let $(1, 2)=f_1$, $(3, 4)=f_2$, $(3, 4, \cdots, n)=f_3$, $(1, n)=f_4$, $(2, 3)=f_5$ and $(1, n)(2, 3)=f_6$ $(n\geq 4)$. Let $\phi\in\Sigma_n$ map 2 onto 1 and leave every other element of I unchanged. Let (Q) be the set of relations: $f_1\phi=f_2\phi f_2=f_3\phi f_3^{-1}=(\phi f_4)^2=\phi$; $(f_5\phi)^2=\phi f_5\phi=(\phi f_5)^2$; $(\phi f_6)^2=(f_6\phi)^2$. Let M be any irreducible set of generators of σ_n and let (P) be a set of defining relations for σ_n. When the f_i are expressed as words in the elements of M, the author shows that then the set $(P)\cup(Q)$ is a set of defining relations for Σ_n in terms of the set of generators $M\cup\{\phi\}$. The proof proceeds by showing that each element of Σ_n can be written in the canonical form

$$(1, i_1)(2, i_2)\phi(1, i_3)(2, i_4)\phi\cdots(1, i_{2p-1})(2, i_{2p})\phi g,$$

where $i_1<i_2<i_4<\cdots<i_{2p}$; $i_{2k-1}<i_{2k}$, i_{2k-1}, $i_{2k}\neq 2$ and $i_{2m-1}\neq i_{2k}$ for $2\leq k, m\leq p$; $p\geq 0$ and $g\in\sigma_n$. Such an expression for an element of Σ_n is unique except for the choice of the factor g, and any element of Σ_n can be reduced to this form by means of the relations $(P)\cup(Q)$.
G. B. Preston (Shrivenham)

89:

Clifford, A. H. Completion of semi-continuous ordered commutative semigroups. Duke Math. J. **26** (1959), 41–59.

An ordered commutative semigroup (o. c. s.) is a linearly ordered set which is also a commutative semigroup satisfying $a<b$ implies $a+c\leq b+c$ where $+$ is the semigroup operation. By a normal completion of an ordered set the author means the usual Dedekind completion. The purpose of the paper is to investigate semigroups which are normal completions of an o. c. s. An o. c. s. is defined to be lower semi-continuous (l. s. c.), upper semi-continuous (u. s. c.) or continuous if $+$ is respectively l. s. c., u. s. c. or continuous in the order topology. The author shows, among other results, that an l. s. c. o. c. s. (his notation) possesses a unique l. s. c. normal completion [see also Krishnan, Bull. Soc. Math. France **78** (1950), 235–263; MR **13**, 201]. A continuous o. c. s. may, in general, have many normal completions. The author gives conditions for uniqueness and also examples to illustrate the various possibilities. Finally, normal completions of naturally ordered [Amer. J. Math. **76** (1954), 631–646; MR **15**, 930] o. c. s.'s are studied, and conditions are given for the existence of naturally ordered completions of naturally ordered o. c. s.'s.
Haskell Cohen (Baton Rouge, La.)

90:

Stolt, Bengt. Zur Axiomatik des Brandtschen Gruppoids. Math. Z. **70** (1958), 156–164.

An analysis with respect to independence is given for axioms for Brandt gruppoids [H. Brandt, Math. Ann. **96** (1926), 360–366], with primary attention to separating assertions of existence and uniqueness.
R. C. Lyndon (Ann Arbor, Mich.)

91:

Kattsoff, L. O. The independence of the associative law. Amer. Math. Monthly **65** (1958), 620–623.

Three elementary examples are given for an algebraic system, which satisfies all postulates for an abelian group except the associative law. *Št. Schwarz* (Bratislava)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 691, 694.

92:

Harish-Chandra. Spherical functions on a semisimple Lie group. II. Amer. J. Math. **80** (1958), 553–613.

The author continues the work toward an explicit Plancherel formula for spherical functions which he began in part I [Amer. J. Math. **80** (1958), 241–310; MR **20** #925]. He succeeds in establishing the desired formula for all members of a certain subspace $I_0'(G)$ of the space $I_2(G)$ of all square integrable spherical functions on the underlying semisimple Lie group G. A final solution of the problem depends upon showing that $I_0'(G)$ is dense in $I_2(G)$. The author shows how this would follow from the truth of two conjectures which, however, he says he does not know how to prove. These conjectures are too technical to be stated here. As in I the arguments are long and complicated and involve a detailed analysis of the asymptotic behavior of the coefficients and solutions of differential equations on Lie groups. Extensive use is made of a method of reducing questions about spherical functions on a group G to similar questions about spherical functions on a proper subgroup. *G. W. Mackey* (Cambridge, Mass.)

93:

Auslander, Louis. Some compact solvmanifolds and locally affine spaces. J. Math. Mech. **7** (1958), 963–975.

In this paper the author gives a sufficient condition (not

necessary) for an abstract group to be the fundamental group of a compact solv-manifold. The following are proved: "Let Z^n denote the direct product of n infinite cyclic groups, and Γ be an extension of a Z^s by a Z^t. Then Γ can be imbedded, as a closed subgroup, in a connected solvable Lie group S such that S/Γ is compact. Moreover S is an extension of a complex vector space C^s by a real vector space R^t, and the manifold S/Γ has a complete affine connection without curvature and torsion."

H. C. Wang (Evanston, Ill.)

94:

Hartman, S.; et Hulanicki, A. Sur les ensembles denses de puissance minimum dans les groupes topologiques. Colloq. Math. **6** (1958), 187–191.

The authors study the possible cardinalities of dense and isolated subsets of topological groups under the assumption of the generalized continuum hypothesis. Sample result: An infinite compact abelian group of cardinality $\leqq 2^{2^m}$ has a dense subset of cardinality $\leqq m$.

K. deLeeuw (Stanford, Calif.)

95:

Freudenthal, Hans. Zur Klassifikation der einfachen Lie-Gruppen. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 379–383.

Based on Weyl's results, Dynkin [Mat. Sb. (N.S.) **18** (**60**) (1946), 347–352; MR **8**, 133] has proved that a complex simple Lie algebra G is characterized by its simple roots, and thus reduced the classification of G to the classification of Π-systems. (By a Π-system, we mean a set of linearly independent vectors $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the euclidean space R^n such that the angle between any two of them is either 90°, 120° or 135° and that the set cannot be decomposed onto the union of 2 mutually orthogonal subsets). Dynkin's method of determining Π-systems was presented by P. Cartier [Seminaire "Sophus Lie" de l'École Normale Superieure, 1954–55, Secrétariat mathématique, Paris, 1955; MR **17**, 384] in a very clear manner. In this paper the author gives modifications which lead to a still more lucid presentation of this method.

H. C. Wang (Evanston. Ill.)

96:

Ellis, Robert. Distal transformation groups. Pacific J. Math. **8** (1958), 401–405.

A group of homeomorphisms G of a space X onto itself is called distal if, given any three points x, y, z in X and any filter $\mathscr{F}$ on G, then $x\mathscr{F}\to z$, and $y\mathscr{F}\to z$ implies $x=y$. (For this and other terms used here, cf. Gottschalk and Hedlund, *Topological dynamics* [Amer. Math. Soc. Colloq. Publ. no. 36, Providence, R.I., 1955; MR **17**, 650].) Theorem 1: Let X be a Hausdorff space and G a group of homeomorphisms of X onto X such that $\overline{xG}$ is compact for all $x\in X$; then the following statements are pairwise equivalent: (1) The closure T of G in X^X is a compact group; (2) for every cardinal $\mathfrak{a}>0$, G is pointwise almost periodic on $X^{\mathfrak{a}}$; (3) there exists a cardinal $\mathfrak{a}>1$ such that G is pointwise almost periodic on $X^{\mathfrak{a}}$; (4) the group G is distal. For abelian G's generated by some compact neighbourhood of the identity, the author proves theorem 2: Let X be locally compact zero-dimensional, let G be distal, and let $\overline{xG}$ be compact for all $x\in X$; then G is equicontinuous. Further results are obtained for groups with compact zero-dimensional orbit closures.

L. W. Green (Minneapolis, Minn.)

97:

Ehresmann, Charles. Sur les pseudogroupes de Lie de type fini. C. R. Acad. Sci. Paris **246** (1958), 360–362.

Sur une variété différentiable V_n un "pseudogroupe de Lie Γ, de type fini", est l'ensemble des solutions d'une sous-variété ϕ^r du groupoïde Π^r des r-jets inversibles de V_n dans V_n, cette sous-variété ϕ^r jouissant des propriétés suivantes: (1) ϕ^r est un sous-groupoïde de Π^r; (2) l'application canonique de ϕ^r dans Π^{r-1} est localement biunivoque (système de Mayer-Lie); (3) ϕ^r est complètement intégrable.

La Note est en grande partie consacrée à la démonstration du théorème suivant: le plus grand groupe de transformations contenu dans un pseudogroupe de Lie de type fini est un groupe de Lie. La démonstration utilise essentiellement les propriétés des feuilletages et des espaces étalés; C. Ehresmann introduit notamment la notion de groupe d'holonomie d'un feuilletage. Un résultat classique d'E. Cartan est utilisé: un noyau de sous-groupe fermé dans un noyau de groupe de Lie est un noyau de groupe de Lie. Le théorème se généralise dans le cas où la condition (3) n'est pas vérifiée; on a: étant donnée une G-structure γ dont le groupe structural $G\subset L_n$ est tel qu'on puisse associer canoniquement une connexion affine à toute G-structure, le groupe des automorphismes de γ est un groupe de Lie.

En utilisant le théorème de stabilité de Reeb, le théorème suivant est encore démontré: si V_n est compacte, si le groupe d'holonomie du feuilletage de ϕ^r se réduit à l'identité et si ϕ^r est connexe, alors le pseudogroupe Γ est déduit par localisation d'un groupe de Lie. On définit les variétés "complètes" pour lesquelles ce théorème est encore vrai.

{Remarque: Ces théorèmes ont été également démontrés en utilisant des méthodes différentes par l'auteur de ce résumé [P. Libermann, C. R. Acad. Sci. Paris **246** (1958), 41–43, 531–534, 1365–1368; MR **20** #2763, #2764, #2765].}

P. Libermann (Rennes)

98:

Kanno, Tsuneo. A note on the Lie algebras of algebraic groups. Tôhoku Math. J. (2) **10** (1958), 262–273.

Chevalley's theory of replicas for algebraic groups of matrices over a field of characteristic zero [*Théorie des groupes de Lie*, Tome II, Hermann, Paris, 1951; MR **14**, 448] is here extended to arbitrary algebraic groups defined over a field of characteristic zero. The Lie algebra of the intersection of two algebraic subgroups of such an algebraic group G is the intersection of their Lie algebras, so, calling a subalgebra of the Lie algebra of G algebraic if it corresponds to an algebraic subgroup of G, each element of the Lie algebra of G is contained in a unique minimal algebraic subalgebra, whose elements are called the replicas of the given element. A subalgebra of the Lie algebra of G is then algebraic if and only if it contains all the replicas of each of its elements.

M. Rosenlicht (Evanston, Ill.)

FUNCTIONS OF REAL VARIABLES

See also 255, 256, 258, 261, 265.

99:

Knobloch, Hans-Wilhelm. Integrale über Funktionen aus multiplikativen Klassen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1957**, 25–64 (1958).

$f(x)$ is called a multiplicative function of x if f'/f is a

rational function of x. An extensive arithmetical and analytical theory of multiplicative classes of functions was developed by R. König, H. Schmidt, and H. Röhrl [Röhrl, Math. Ann. **123** (1951), 53–75; **124** (1952), 187–218; **125** (1953), 448–466; MR **13**, 224, 637; **15**, 31; gives references to earlier work]. In this theory f'/f is a rational function of x over the field of complex numbers.

In the present paper an analogous theory is developed for multiplicative functions $f(x)$ for which f'/f is a rational function of x over an arbitrary field of characteristic 0, and, in particular, over a field of analytic functions of a second complex variable, u. This theory is applied to integral transformations $t(x)\to T(u)$, where

$$T(u) = \int_C K(x, u)t(x)dx,$$

the kernel $K(x, u)$ belongs to a multiplicative class, and C is a path which is closed on the Riemann surface of K. In particular, the periods $\int_C K(x, u)dx$ are studied when K ranges over a multiplicative class and C over the set of closed paths. The results are applied to a discussion of integral representations of solutions of a linear ordinary differential equation with linear coefficients, and it is pointed out that the results concerning complementary systems of periods are a source of numerous relations involving special functions.

A. Erdélyi (Pasadena, Calif.)

100:

Zaubek, Othmar. Über einige mit dem Zwischenwertsatz verwandte Entwicklungen. Math. Z. **69** (1958), 351–362.

In this paper generalizations of the intermediate value theorem for real continuous functions are proved. Let T and T^* be topological spaces, $f|A$ a function whose domain $A\subset T$ and whose range belongs to T^*; let a be either a point or a point of accumulation of A. Then the author calls $h\in T^*$ a "Hüllenwert $h(f|A\,;\,a)$" or a "Häufungswert $h'(f|A\,;\,a)$" of $f|A$ at a if to every neighborhood V of h in T^* and to every neighborhood U or to every reduced neighborhood U', respectively, of a in T there is at least one point $b\in AU$ or $\in AU'$ such that $f(b)\in V$. The set of all $h(f|A\,;\,a)$ (for fixed a) or of all $h'(f|A\,;\,a)$ is called the "Hülle $H(f|A\,;a)$" or the "reduced Hülle $H'(f|A\,;\,a)$" (sometimes called "set of accumulation values") of $f|A$ at a, respectively. Then the author proves the "Zwischenhüllenwertsatz": If $f|A$ is a real-valued function with domain A in a topological space T and if B is a set with the following properties: (1) B is the intersection of A with an open set O; (2) there is a connected set Z with $B\subseteqq Z\subseteqq \bar{A}O$ (where $\bar{A}$ is the closure of A); (3) at every point $x\in Z$, the "Hülle" $H(f|A\,;\,x)$ is connected; then every real number r between the infimum g and the supremum G of the function $f|B$ (i.e., the function $f|A$ reduced to B) is a "Hüllenwert" of $f|B$.

Another definition of the author is the following: Let $f|A$ be defined on A and let B be an arbitrary point set. Then the set of those points z of the range T^* of $f|A$ to which there is at least one $b\in B$ such that $z\in H'(f|A\,;\,b)$, and for every $x\in AB$ we have $f(x)\neq z$, is called the "Häufungswerteüberschuss $\ddot{U}(f|A\,;\,B)$ of $f|A$ for the set B. Then one obtains the "generalized intermediate value theorem": With the notations and assumptions of the "Zwischenhüllenwertsatz", $f|B$ takes on all the values between its inf and sup except for the "Häufungswerteüberschuss" of $f|B$ for the set Z. The author proves also a few other theorems of an analogous character (after having introduced some more definitions).

A. Rosenthal (Lafayette, Ind.)

101:

Svetlanov, A. V. Principles of generalized differential-integral calculus. Dokl. Akad. Nauk SSSR **123** (1958), 981–983. (Russian)

Par la dérivée généralisée d'ordre n (n réel ou complexe) d'une fonction $u(x)$ continue et indéfiniment dérivable au sens ordinaire on entend d'après l'auteur l'opération

$$(*)\qquad D^n u(x) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1)x^{k-n}}{k!\Gamma(n-k+1)\Gamma(k-n+1)}\,u^{(k)}(x)$$

supposée convergente. Si $D^nF(n, x)\equiv 0$ alors on a $F(n, x)\equiv \sum_{k=1}^{\infty} c_k x^{n-k}$ ($n-k\neq -N$; N un entier positif). Si (*) a lieu pour tout γ complexe et fini, alors l'ensemble des opérations $D=\{D^\gamma\}$ forme un groupe multiplicatif avec l'élément neutre $D^0(x)=u(x)$. On indique aussi les relations entre la dérivée généralisée d'ordre n et des semi-groupes analytiques.

M. Tomić (Belgrade)

102:

Krylov, A. L. On a necessary and sufficient condition which can be used as a test that a given function belongs to the class $W_p^{(1)}$ of Sobolev. Dokl. Akad. Nauk SSSR **121** (1958), 795–796. (Russian)

The author gives a definition of p-variation of a function of n ($\geqq 2$) variables and shows that for $p>n$ the functions of the class $W_p^{(1)}$ of Sobolev [Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565] are exactly the functions of finite p-variation.

M. M. Day (Urbana, Ill.)

103:

Il'in, V. P. Some functional inequalities of the type of theorems of imbedding. Dokl. Akad. Nauk SSSR **123** (1958), 967–970. (Russian)

D is a subset of n-space which has the property that each of its points is the vertex of a spherical sector of fixed shape and of radius at least H. D_m, $[D_m]_{n-m}^d$ are, respectively, sets $x_i=a_i$, $|x_i-a_i|\leqq d$, $i=m+1, \cdots, n$, a_i some fixed numbers. f is a function with continuous derivatives up to lth order in D, such that

$$\left[\int_D |f|^p dv_n\right]^{1/p} \leqq A,$$

$$\max_{D_m}\left[\int_{[D_m]_{n-m}^d}\left(\sum\left|\frac{\partial^l f}{\partial x_{i_1}\cdots\partial x_{i_l}}\right|^2\right)^{p/2} dv_n\right]^{1/p} \leqq Md^{\alpha_m},$$

where dv_n is the volume element in n-space, α_m a sequence of integers decreasing to $\alpha_n=0$, $\alpha_m\leqq (n-m)/p$. Then, for an integer k, $0\leqq k\leqq l-1$, inequalities are given for the values of $\varphi(x)=\partial^k f/\partial x_{i_1}\cdots\partial x_{i_k}$ and for $\int_{D_m}|\varphi|^q dv_m$, where q is an exponent in a certain range; the inequalities compare the quantities mentioned with rather complicated powers of an arbitrary number $h<H$. Further inequalities compare the maximum of $|\varphi(x)-\varphi(y)|/|x-y|^\lambda$, where λ is an exponent less than 1, with expressions in terms of h. All results are stated without proof.

J. L. B. Cooper (Cardiff)

104:

v. Krbek, F. Belebung der Analysis. Arch. Math. **9** (1958), 433–435.

The author gives three examples, each of independent pedagogical interest, and also important in various aspects of analysis. The first example describes some properties of f and F, where f is monotone increasing in $[a, b]$ and $F(y) = \sup\{x : f(x) \leq y\}$. The second notes that $x/\ln x$ is monotone increasing for $x > e$ and rather casually refers to the prime number theorem. The third is concerned with a form of the Poincaré-Carathéodory Wiederkehrsatz.

R. D. James (Vancouver, B.C.)

105:

Berg, Lothar. Einführung des Differentialquotienten vom Standpunkt der praktischen Mathematik. Wiss. Z. Hochschule Elektrotechn. Ilmenau **3** (1957), 189–190.

The author recommends, for pedagogical purposes and for explaining Newton's method of solving equations, the definition of the derivative $f'(x_1)$ as a number m such that

$$f(x) = f(x_1) + m(x - x_1) + \rho(x, x_1)(x - x_1),$$

with $\rho \to 0$ as $x \to x_1$.

R. P. Boas, Jr. (Evanston, Ill.)

106:

de Vito, Luciano. Su un esempio di funzione continua senza derivata. Enseignement Math. (2) **4** (1958), 281–283.

It was proved by de Rham [same Enseignement **3** (1957), 71–72; MR **19**, 20] that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k}|2^k x - [2^k x + \tfrac{1}{2}]|,$$

where $[y]$ denotes the greatest integer that is not greater than y, is continuous and nowhere differentiable on the x-axis. In the present article it is proved that $f(x)$ is also Hölderian for arbitrary index α with $0 < \alpha < 1$. Some further conclusions are drawn.

107:

Glaeser, Georges. Étude de quelques algèbres tayloriennes. J. Analyse Math. **6** (1958), 1–124; erratum, insert to **6** (1958), no. 2.

Let K be a compact set in n-space R^n, F a closed subset of K, and m a positive integer. Let $\mathfrak{D}^m(K)$ be the algebra of m-times continuously differentiable functions on K and $J^m(F)$ the ideal consisting of those functions which are 0 on F together with all partial derivatives of orders $1, \cdots, m$. Let $W^m(F) = \mathfrak{D}^m(K)/J^m(F)$ (Whitney algebra). When provided with a suitable norm $W^m(F)$ is a Banach algebra. One of the main objectives of this long paper is to relate the structure of $W^m(F)$ to local properties of the set F. For this purpose the author introduces the following variant of the notion of paratingent to a set. By class m local immersion of F at a point $A \in F$ is meant an m-times continuously differentiable manifold containing F intersected with a neighborhood of A. Let $\mathrm{ptgl}^m(A)$ [paratingent linéarisé d'ordre m] denote the intersection of the tangent spaces at A of all class m local immersions of F. Then $\mathrm{ptgl}^1(A)$ is the smallest field of contact elements on F which contains the paratingent to F at A for every $A \in F$ and is an upper semicontinuous function of A. Theorem: $W^m(F)$ is semi-simple if and only if $\mathrm{ptgl}^1(A)$ has dimension n for every $A \in F$. Theorem: $W^m(F)$ is a Wedderburn algebra (i.e., $W^m(F)$ is the direct sum of its radical and a subalgebra) if and only if F is the union of a finite number of closed sets on each of which the dimension d_m of $\mathrm{ptgl}^m(A)$ is constant and equal to d_1. A chapter is devoted to algebras of m-times continuously differentiable functions such that the mth derivatives satisfy a condition of Hölder type. The final chapter generalizes Šilov's work on continuous sums of algebras of finite dimension [Mat. Sb. (N.S.) **27** (**69**) (1950), 471–484; MR **12**, 618].

W. H. Fleming (Providence, R.I.)

108:

Menger, Karl. Multiderivatives and multi-integrals. Amer. Math. Monthly **64** (1957), no. 8, part II, 58–70.

The author and S. S. Shü in a very condensed note [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 591–595; MR **17**, 21], referred to below as MS, defined the derivative polynomial $\mathsf{D}_g{}^m f(\mathrm{A})$ ("multiderivative of order $m = 1, 2, \cdots$") of a function f with respect to a function g and a finite system A of numbers under suitable assumptions on f, g and A and produced infinite expansions of f involving such derivatives. The present paper does not provide the missing proofs: it supplies illuminating motivations and examples in § 1–5 and some new material in § 6–8. Variable x and function f are first assumed to be real to allow a geometrical interpretation, a and b are two distinct numbers interior to the domain of definition E of f. The biderivative ${}_2\mathsf{D}f(a, b)$ ($\mathsf{D}f(a, b)$ in MS) results from the comparison of f with the gauge polynomial $g(x) = (x-a)(x-b)$ in neighborhoods of a and b, namely ${}_2\mathsf{D}f(a, b) = \lim\left((f[a', b'] - f[a, b])/(g[a', b'] - g[a, b])\right)$ when $a' \to a$, $b' \to b$ such that the denominator be a constant function, that is to say, in this case $a' + b' = a + b$. The case $a = b$ is treated similarly. Quite generally, if A is a system of n numbers interior to E,

$${}_n\mathsf{D}f(\mathrm{A}) = \lim\left((f[\mathrm{A}'] - f[\mathrm{A}])/(g[\mathrm{A}'] - g[\mathrm{A}])\right)$$

as $\mathrm{A}' \to \mathrm{A}$, where $g(x)$ is the product of the factors $x - a$ for $a \in \mathrm{A}$, and A' such that $g[\mathrm{A}']$ be a constant polynomial, then the n-tuples A' depend on one parameter only. The second order biderivative ${}_2\mathsf{D}^2 f(a, b)$ (shorter $\mathsf{D}^2 f(a, b)$) of f at (a, b) is defined as $\lim\left((f[a', b'] - f[a, b] - {}_2\mathsf{D}f(a, b) g[a', b'])/(g^2[a', b']/2!)\right)$ as $(a', b') \to (a, b)$, $a' + b' = a + b$. The preceding definitions can be readily transposed or extended to any gauge function g, any finite system A, and any order whether in the real or complex field. Different gauges may yield the same derivative, for example if $\mathrm{A} = (a, b)$, $a \neq b$; $g_1(x) = x^2$; $g_2(x) = (x-a)(x-b)$; $g_3(x) = (a-b)(x-a)$ if $x \leq \frac{1}{2}(a+b)$, $g_3(x) = (b-a)(x-b)$ if $x \geq \frac{1}{2}(a+b)$. Under suitable assumptions on f, the Riemann integral in t,

$$-\int_0^1 \mathsf{D}f(a - ut, b + ut)\, d((a - ut)(b + ut)),$$

where $u = b' - b = a - a'$, exists and is $= f[a', b'] - f[a, b]$.

Chr. Pauc (Nantes)

109:

Pall, Gordon. Generalized Jacobi expansions and corresponding derivatives. Amer. Math. Monthly **64** (1957), no. 8, part II, 71–78.

The Menger-Shü derivatives $\mathsf{D}^r f(\mathrm{A})$, $\mathsf{D}_g{}^r f(\mathrm{A})$ [see preceding review] are, for f and g fixed, functions of x and of the n points of A. $P(z)$ denoting a polynomial of degree n in the complex field,

$$f(z) = A_0(z) + A_1(z)P(z) + \cdots + A_r(z)P(z)^r + \cdots$$

the corresponding Jacobi expansion, then $r!A_r = \mathsf{D}^r f(\mathrm{A})$ for A = system of zeros of P. In § 2 of the present paper is examined a simple way of developing the theory of Jacobi series:

$$w = P(z) = z^2 - 2az + b, \quad c = b - a^2 \neq 0,$$

W = interior of the circle of centre 0 and radius $|c|$ in the w-plane, $z_1 = p_1(w)$ and $z_2 = p_2(w)$ are the solution-functions of $w = P(z)$ on W, which are regular and map it one-to-one onto Z_1 and Z_2, respectively. Any function $f(z)$ regular on $Z = Z_1 + Z_2$ can be written in the form $f(z) = a_1(w) + za_2(w)$, $a_1(w)$ and $a_2(w)$ being uniquely determined and regular on W and $w = P(z)$. Then $f(z) = \sum_{r=0}^{\infty} A_r(z)P(z)^r$ on Z where $A_r(z) = a_1^{(r)}(0) + za_2^{(r)}(0)$; in particular, $A_1(z) = a_1'(0) + za_2'(0)$. The idea of a more general basic decomposition of f lies behind § 3 and 4: The domain W of the w plane is mapped one-to-one onto Z_i, $i = 1, 2, \cdots, n$, by a function p_i, the points $z_i = p_i(w)$ being distinct. There exists a function h ("gauge function") on $Z = \bigcup Z_i$ to W such that $h(p_i(w)) = w$ on Z_i. By an "interpolation basis" is meant a sequence of n functions $q_1, \cdots, q_n$ on Z such that every function f on Z may be written in a unique manner in the form $f(z) = a_1(w)q_1(z) + \cdots + a_n(w)q_n(z)$ where $w = h(z)$. Assuming the existence of an interpolation basis $q_1, \cdots, q_n$ and suitable differentiability properties for f, h, p_i and q_i, the "h-derivative" $\mathsf{D}_h f$ is defined on Z as a one-place function by $\mathsf{D}_h f(z) = a_1'(h(z))q_1(z) + \cdots + a_n'(h(z))q_n(z)$. If $f = \psi(h)$ and $q_1(z) = 1$ on Z, then $\mathsf{D}_h f = \psi'(h)$. An expression similar to a divided difference formula is given; it involves z and $z_i = p_i(h(z))$. Analogues of Rolle's and mean-value theorems when $h(x) = x^2 - rx$ are discussed in § 5. {Remark by the reviewer: The term "h-derivative" of f seems justified only in so far as the mappings p_i are completely determined by h, which is the case in the examples where h is a polynomial and the p_i are all determinations of the inverse mapping of h on W.} *Chr. Pauc* (Nantes)

MEASURE AND INTEGRATION

See also 104, 664.

110:

Bell, C. B. On the structure of stochastic independence. Illinois J. Math. **2** (1958), 415–424.

Let $\{(Y, \mathfrak{M}_t, \mu_t)\}$ $(t \in \mathfrak{H})$ be an arbitrary at-least-countable family of probability σ-measure spaces. Consider the four conditions: (C_0) The $\{\mathfrak{M}_t\}$ are σ-independent, i.e., $\bigcap_1^\infty A_{t_i} \neq \emptyset$ whenever $\emptyset \neq A_{t_i} \in \mathfrak{M}_{t_i}$ for all i and $t_i \neq t_j$ for $i \neq j$; (C_1) the $\mathfrak{M}_t$ are stochastically independent with respect to the μ_t, i.e., there exists a σ-measure μ (called the stochastic extension of the μ_t) on $\mathfrak{S}(\mathfrak{M}_t)$, the least σ-algebra containing all $\mathfrak{M}_t$, such that $\mu(\bigcap_1^n A_{t_i}) = \prod_1^n \mu_{t_i}(A_{t_i})$ whenever $A_{t_i} \in \mathfrak{M}_{t_i}$ for $1 \leq i \leq n$ and $t_i \neq t_j$ when $i \neq j$; (C_2) the $\mathfrak{M}_t$ are almost σ-independent with respect to the $\{\mu_t\}$, i.e., $\bigcap_1^\infty A_{t_i} \neq \emptyset$, whenever $\mu_{t_i}(A_{t_i}) \neq 0$ and $\emptyset \neq A_{t_i} \in \mathfrak{M}_{t_i}$ for all i; and $t_i \neq t_j$ for $i \neq j$; (C_3) the $\{\mathfrak{M}_t\}$ are quasi-σ-independent with respect to the $\{\mu_t\}$, i.e., $\bigcap_1^\infty A_{t_i} \neq \emptyset$, whenever $\prod_1^\infty \mu_{t_i}(A_{t_i}) \neq 0$ and $\emptyset \neq A_{t_i} \in \mathfrak{M}_{t_i}$ for all i and $t_i \neq t_j$ for $i \neq j$. It is known that $(C_0) \to (C_2) \to (C_3)$ and $(C_0) \to (C_1) \to (C_3)$. This paper shows that $(C_3) \to (C_1)$, $(C_3) \not\to (C_1)$, $(C_1) \not\to (C_2)$ and $(C_2) \not\to (C_0)$.

S. Sherman (Philadelphia, Pa.)

111:

Perkal, J. On the length of empirical curves. Zastos. Mat. **3** (1958), 258–284 (2 inserts). (Polish. Russian and English summaries)

The author develops in greater detail his concept of ε-length [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 399–403; MR **18**, 384], discusses the operation of his longimeter, and produces some empirical data about the probability distribution of assessments of ε-lengths by this method. Results of the author's paper "Sur les ensembles epsilon convexes" [Colloq. Math. **4** (1956), 1–10; MR **17**, 999] are applied to the problem of "generalising curves", which is well known in geography: Given a length ε and a curve X locally dividing the plane into two regions A and B, the ε-generalisation of A is the smallest 2ε-convex set containing A, and its boundary is described as the ε-generalised boundary of A in B. This and the ε-generalised boundary of B in A are two possible generalisations of X. The lengths of both these generalisations can be evaluated by means of an ε-longimeter. *S. K. Zaremba* (Swansea)

112:

Zink, Robert E. On regular measures and Baire functions. Monatsh. Math. **63** (1959), 19–23.

Continuing his earlier study [Duke Math. J. **24** (1957), 127–135; MR **19**, 21] of outer regularity of measures defined by integrals, the author establishes, via several lemmas, the following theorem: given a topological space X, an outer regular measure μ on the σ-ring of Borel subsets of X, and a non-negative Borel measurable Baire function f on X, then the measure ν defined for each Borel set E by setting $\nu(E) = \int_E f d\mu$ is outer regular if and only if for each Borel set E over which f is integrable there exists a sequence $\{U_n\}$ of open Borel sets such that f is integrable over each U_n and $E \subset \bigcup_n U_n$.

T. A. Botts (Charlottesville, Va.)

113:

Jeffery, R. L. Generalized integrals with respect to functions of bounded variation. Canad. J. Math. **10** (1958), 617–626.

$\omega(x)$: finite non-decreasing function defined on the closed interval $[a, b]$. C: set on which ω is continuous. $F(x)$: finite function defined and continuous on C with finite right and left limits $F(x+)$, $F(x-)$ at any point $x \in [a, b]$. By "ω-measure" is meant the Lebesgue-Stieltjes measure associated to ω. The difference quotient $\psi(x, h)$ is defined for $x \in [a, b]$, $x + h \in C$ as

$$(F(x+h) - F(x-))/(\omega(x+h) - \omega(x-))$$

if $h > 0$ and $\omega(x+h) - \omega(x-) \neq 0$; as

$$(F(x+h) - F(x+))/(\omega(x+h) - \omega(x+))$$

if $h < 0$ and $\omega(x+h) - \omega(x+) \neq 0$; and as 0 if $\omega(x+h) - \omega(x\pm) = 0$. The ω-derivative $D_\omega F(x) = \lim \psi(x, h)$ as $h \to 0$, provided the limit exists. F is "AC-ω" (on E) if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite set of non-overlapping intervals (x_i, x_i') on $[a, b]$ (with $x_i \in E$, $x_i' \in E$) the inequality $\sum (\omega(x_i'+) - \omega(x_i-)) < \delta$ implies $\sum |F(x_i'+) - F(x_i-)| < \varepsilon$. F is "ACG-ω" on $[a, b]$ if this interval is the union of a denumerable sequence of closed sets on each of which F is AC-ω. Theorem 2: If F is AC-ω on $[a, b]$, if $D_\omega F(x) = f(x)$ except for a set of ω-measure zero and if $a \in C$, $x \in C$, then $F(x) - F(a) = \int_a^x f(x) d\omega$. Theorem 4: If F is ACG-ω on $[a, b]$, if f is an ω-measurable function on

$[a, b]$ and $D_\omega F(x)=f(x)$ except for a set of ω-measure zero, then $F(b+)-F(a-)$ can be determined in a denumerable set of operations. Hints to the proofs: They rest on a Vitali property of intervals relatively to the ω-measure (lemma 2). Theorem 4 is proved by adapting the classical (i.e., $\omega(x)=x$) totalization procedure. {Remarks by the reviewer: (1) As it stands, lemma 2 is incorrect. Here is a counter-example: $a=0$, $b=1$; E consists of the point $x=\frac{1}{2}$; the open intervals $(x, x+h_i)$ are $(1/2, 1/2+1/2^i)$, $i=1, 2, \cdots$; $\omega(x)=0$ when $0\leq x<\frac{1}{2}$; $\omega(x)=1$ when $\frac{1}{2}\leq x\leq 1$. Lemma 2 holds for intervals $[x, x+h_i)$. This alteration does not affect the application p. 621 because H is continuous, but it invalidates the equality p. 625, lines 9–10. The closed interval function $F(x'')-F(x')$, $x'<x''$, is additive for partitions in non-overlapping intervals; $F(x''+)-F(x'-)$ is not for discontinuous ω. (2) From results of A. P. Morse [Trans. Amer. Math. Soc. **61** (1947), 418–442; MR **8**, 571; p. 431] and C. A. Hayes and C. Y. Pauc [Canad. J. Math. **7** (1955), 221–274; MR **17**, 719; p. 256 (Proposition 3.16) and p. 235 (Theorem 1.64)], for the closed interval basis on $[a, b]$, the full differentiation theorem of any (signed) Lebesgue-Stieltjes measure with respect to any ω-measure follows. This theorem includes the information conveyed by theorem 2 above.}

Chr. Pauc (Nantes)

114:

Krickeberg, Klaus. Stochastische Derivierte. Math. Nachr. **18** (1958), 203–217.

The present work is based on three previous papers of the author: (K1) Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955** (1956), 217–279 [MR **18**, 118]; (K2) Trans. Amer. Math. Soc. **83** (1956), 313–337 [MR **19**, 947]; (K3) Math. Z. **66** (1957), 470–486 [MR **19**, 948]. The notations of the MR review of (K1) are adopted below. In (K3), for any stochastic process (f_σ) with a directed index set, lower and upper stochastic limits $\sigma \lim\inf f_\sigma$ and $\sigma \lim\sup f_\sigma$ were defined and the Fatou lemmas appropriately generalized. The new tools are now used to achieve a considerable advance in the Burkill theory of cell functions. $\mathfrak{T}=(\mathfrak{T}, \sqsubset)$: $\sqsubset$-directed set of cell partitions $\mathscr{T}$ of E. $\mathscr{K}=\bigcup\mathfrak{T}$. For $K\in\mathscr{K}$, $\mathfrak{T}_K$ denotes the set of the cell partitions of K which can be extended to a partition of $\mathfrak{T}$. $\underline{\psi}$, $\bar{\psi}$: Burkill-Kolmogorov $\sqsubset$-integrals of ψ. $\underline{D}\psi$, $\bar{D}\psi$: essential (or algebraic) $\sqsubset$-derivatives of ψ. § 1. The stochastic derivatives $\sigma\underline{D}\psi$ and $\sigma\bar{D}\psi$ are defined as $\sigma \lim\inf D(\psi, \mathscr{T})$ and $\sigma \lim\sup D(\psi, \mathscr{T})$, respectively. Condition **E**: To any $\mathscr{T}\in\mathfrak{T}$ and any disjoint set $\mathscr{I}$ of $\mathscr{T}$-fine cells there exists $\mathscr{Z}\in\mathfrak{T}$ with $\mathscr{I}\subseteq\mathscr{Z}$ and $\mathscr{T}\sqsubset\mathscr{Z}$. Theorem 1.3: If $\mathfrak{T}$ fulfills **E** and ψ is of bounded variation and absolutely continuous, then $\sigma\underline{D}\psi$ and $\sigma\bar{D}\psi$ are μ-summable and their μ-integrals $\underline{I}(\psi, E)$ and $\bar{I}(\psi, E)$ are equal to $\underline{\psi}(E)$ and $\bar{\psi}(E)$, respectively. Sketch of the proof: The existence of the μ-integrals and the inequalities $\underline{I}(\psi, E)\leq\underline{\psi}(E)$, $\bar{I}(\psi, E)\geq\bar{\psi}(E)$ follow from the stochastic Fatou lemmas. The inequality $\bar{I}(\psi, E)\leq\bar{\psi}(E)$ has to be proved. To any $\gamma<\bar{I}(\psi, E)$ there corresponds a strict minorant f of $\sigma\bar{D}\psi$ with μ-integral $>\gamma$. From the definition of $\sigma\bar{D}\psi$ follows, for any $\mathscr{R}\in\mathfrak{T}$, the existence of a sequence $\mathscr{X}_1, \cdots, \mathscr{X}_i, \cdots$ in $\mathfrak{T}$ such that $\mathscr{R}\sqsubset\mathscr{X}_1\sqsubset\mathscr{X}_2\sqsubset\cdots$ and $E=\bigvee\{D(\psi, \mathscr{X}_i)>f\}$. $\mathscr{L}_i$ denotes the maximal subset of $\mathscr{X}_i$ satisfying (a) $(L\in\mathscr{L}_i)\to(\psi(L)/\mu(L)>f[L])$, and (b) the $\bigvee\mathscr{L}_i$ are disjoint. Thus $E=\bigvee\mathscr{L}$, where $\mathscr{L}=\bigcup\mathscr{L}_i$. $\mathscr{L}$ is then replaced by a finite subset $\mathscr{I}$ which, because of **E**, is extended to an $\mathscr{R}$-fine partition $\mathscr{Z}\in\mathfrak{T}$ for which $\psi(\mathscr{Z})>\gamma-2\varepsilon$, $\varepsilon>0$ being controlled because of the absolute continuity of ψ and the μ-integral of f. Hence $\bar{\psi}(E)\geq\gamma$, $\bar{\psi}(E)\geq\bar{I}(\psi, E)$. § 2. Condition **P**: Every partition of $\mathfrak{T}_K$ is finite. Theorems 2.2 and 2.7: If **E** and **P** hold and ψ has bounded variation, then $\underline{\psi}|\mathscr{K}$ and $\bar{\psi}|\mathscr{K}$ are additive and of bounded variation, $\sigma\underline{D}\psi$ and $\sigma\bar{D}\psi$ exist and are equal to $\sigma\underline{D}\psi$ and $\sigma\bar{D}\psi$ respectively. § 3. Theorems 3.1 and 3.3: Under condition **E** the validity of the equality $\bar{D}\psi=\sigma\bar{D}\psi$ for any ψ implies the strong Vitali property **V**. The converse is true without postulating **E**. In § 4 is introduced another "direction" $\preccurlyeq$ in $\mathfrak{T}$ and inequalities between stochastic $\preccurlyeq$- and $\sqsubset$-derivatives are given. § 6 deals with the relativization to a cell K of $\preccurlyeq$- and $\sqsubset$-derivatives and BK-integrals defined by means of $\mathfrak{T}_K$ instead of $\mathfrak{T}$.

Chr. Pauc (Nantes)

115:

Bertolini, Fernando. Le funzioni misurabili di punto (d'ultrafiltro) e la derivazione delle funzioni d'insieme (di soma) nella teoria algebrica della misura. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 163–201.

The paper falls into two parts, the first part being an elaboration of some parts of Caratheodory's *Mass und Integral und ihre Algebraisierung* [Birkhäuser Verlag, Basel-Stuttgart, 1956; MR **18**, 117] and a continuation of a previous paper [Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 225–247; MR **20** #2418], and the second dealing with derivatives of measures with respect to a measure. A Caratheodory lattice $\mathfrak{R}$ is defined as a lattice of somas, containing the null element, and is distributive, relatively complemented and closed under denumerable interference: $\sqcap R_k=\sup[R\subset R_k,\ k=1, \cdots, n, \cdots]$. With Caratheodory, a transformation on $\mathfrak{R}$ to $\mathfrak{S}$ is a homomorphism if it transforms null into null, and leaves order, finite union, denumerable interference and difference invariant. $\mathfrak{N}$ is a σ-ideal if it is closed under denumerable conjunction: $\sqcup R_k=\inf[R\supset R_k]$, and inverted order: R' in $\mathfrak{N}$ and $R''\subset R'$ implies R'' in $\mathfrak{N}$.

A homomorphism τ defines the σ-ideal $\mathfrak{N}=[R\ni\tau R=0]$. $\mathfrak{R}/\mathfrak{N}$, the class of equivalence classes under $\mathfrak{N}$, defines the homomorphism $\rho_\mathfrak{N}$ on $\mathfrak{R}$ to $\mathfrak{R}/\mathfrak{N}$ where $\mathfrak{N}=[R\ni\rho_\mathfrak{N}R=0]$. An ultrafilter u is a filter of somas maximal relative to the filter property and satisfies the conditions: u is not empty; 0 is not in u; $R_1\cap R_2$ is in u if and only if both R_1 and R_2 are in u; $R_1\cup R_2$ is in u if and only if R_1 or R_2 is in u. For a soma R the set of ultrafilters u such that R belongs to u defines a homomorphism ωR on $\mathfrak{R}$ to the set $U_\mathfrak{R}$ of ultrafilters of $\mathfrak{R}$, and this provides a model of $\mathfrak{R}$ which is a ring of sets. A σ-ideal $\mathfrak{N}$ of $\mathfrak{R}$ determines the ultrafilter set $U_\mathfrak{R}^\mathfrak{N}$ each of which contains no element of $\mathfrak{N}$, and the transformation $\omega_\mathfrak{N}R$.

A measure function $F(R)$ (positive valued, σ-additive), and a σ-ideal $\mathfrak{N}$ give rise to the measure functions $\alpha_\mathfrak{N}F(R)=\inf[F(R'),\ R'\ni\rho_\mathfrak{N}R'=\rho_\mathfrak{N}R]$, the absolutely continuous part of F relative to $\mathfrak{N}$, and $\beta_\mathfrak{N}F(R)=F(R)-\alpha_\mathfrak{N}F(R)$, the singular part of F relative to $\mathfrak{N}$. A real valued function $f(u)$ on the set of ultrafilters $U_\mathfrak{R}$ of $\mathfrak{R}$ is $\mathfrak{R}$-measurable on a soma R_0 if it is defined on R_0 and if for every y, there exists a soma $R(y)\subset R_0$ such that if $f(u)<y$, then u is in $R(y)$ and if $f(u)>y$, then u is not in $R(y)$. The $\mathfrak{R}$-measurable functions are closely related to the "Orts"-functions of Caratheodory.

If $F(R)$ and $M(R)$ are measures, where $M(R)$ is a reduced measure in the sense that $M(R)=0$ implies $R=0$, then $\lim_{R\to u}F(R)/M(R)$ exists for all ultrafilters u of $\mathfrak{R}$, limit being in the Moore-Smith sense on u. The resulting

function of ultrafilters is $\mathfrak{R}$-measurable on all somas of $\mathfrak{R}$, and has the properties: (a) $F(R)=\int_R (dF/dM)dM$ and (b) if $G(R)=\int_R g(u)dM$, then $dG/dM=g(u)$, provided $g(u)$ is an $\mathfrak{R}$-measurable function and $\int_R g(u)dM$ exists for every R of $\mathfrak{R}$.

To obtain results for the case when $M(R)$ is not a reduced measure and $\mathfrak{N}=[R \ni M(R)=0]$, it is possible to define an (improper) derivative in two equivalent ways:

(a) $dF/dM = \lim_{R\to u} F(\rho_{\mathfrak{N}}R)/M(\rho_{\mathfrak{N}}R) = \lim_{R\to u} \alpha_{\mathfrak{N}}F(R)/M(R)$;

and

(b) $dF/dM = \lim_{R\to u} F(R)/M(R)$ for u in $U^{\mathfrak{R}}_{\mathfrak{N}}$.

Then the Lebesgue decomposition formula holds:

$$F(R) = \beta_{\mathfrak{N}}F(R)+\int_R (dF/dM)dM.$$

It is shown that the derivative so defined agrees with the derivative concept of G. Fichera [*Lezioni sulle trasformioni lineari*, vol. I, Inst. Mat. Univ. Trieste, 1954; MR **16**, 715; 395ff], for positive measures, viz. dF/dM gives the maximum value to the linear functional $\int_R g dM$, taken for all $\mathfrak{R}$-measurable non-negative functions g such that $\int_R g dM \leq F(R)$ for all R. Connection is also made with the Lebesgue-Vitali theory of set derivatives.

T. H. Hildebrandt (Ann Arbor, Mich.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 255.

116:

★Лаврентьев, М. А.; и Шабат, Б. В. Методы теории функций комплексного переменного. [**Lavrent'ev, M. A.; and Šabat, B. V. Methods of the theory of functions of a complex variable.**] 2nd ed. revised. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 678 pp. 15.05 rubles.

The first edition [1951] was reviewed in MR **14**, 457. The present second edition contains a few changes in the last four chapters. The authors also state in their preface that the misprints which made the first edition hard to use have now been removed.

117:

★Лунц, Г. Л.; и Эльсгольц, Л. Э. Функции комплексного переменного с элементами операционного исчисления. [Lunc, G. L.; and Èl'sgol'c, L. È. Functions of a complex variable with elements of the operational calculus.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 298 pp. 5.55 rubles.

This book presents the elementary facts in a form suitable for students of technical schools, with applications to electrostatics, hydrodynamics, etc.

118:

Trohimčuk, Yu. Yu. On an hypothesis of Pompeĭyu. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4 (76), 363–367. (Russian)

A function of a complex variable $f(z)$ is called "monogenic relative to a set E", if for $z, z+\Delta z \in E$ there exists

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = f_E'(z).$$

$f(z)$ is "incompletely monogenic" on a domain D if $D=\bigcup_i E_i$ ($i=1, 2, 3, \cdots$), $E_i \cap E_j = \emptyset$ for $i\neq j$ (the author writes Λ for $\emptyset$) and $f(z)$ is monogenic on every E_i (relative to E_i). The hypothesis of Pompeiyu is: a uniformly continuous function $f(z)$ is analytic in D if $f(z)$ is incompletely monogenic in D. The author proves a theorem: If a continuous function $f(z)$ is incompletely monogenic in a domain D, then $f(z)$ is locally analytic in this domain, i.e., there exists an open set O everywhere dense in D, in every point of which $f(z)$ is analytic. The hypothesis of Pompeiyu is then proved if, in addition, $f(z)$ is univalent in D.

B. A. Amirà (Jerusalem)

119:

Trohimčuk, Yu. Yu. A theorem of H. Bohr and its generalizations. Mat. Sb. N.S. **45** (87) (1958), 233–260. (Russian)

A continuous, complex-valued function $w=f(z)$, defined in a domain D, is said to have "constant dilatation" $\rho(z)$ at a point $z\in D$ if the

$$\lim_{h\to 0} \left|\frac{f(z+h)-f(z)}{h}\right|$$

exists, finite or infinite. Bohr [Math. Z. **1** (1918), 403–420] showed that if a continuous mapping $f(z)$ is schlicht together with a non-zero dilatation $\rho(z)$ at each point of D, then one of $f(z)$ and $\overline{f(z)}$ is analytic in D, while Menshov [Bull. Math. Soc. France **59** (1931), 141–182] proved a similar result with other restrictions replacing the non-zero condition on $\rho(z)$. The author shows that an arbitrary continuous function having a constant dilatation at each point of D is analytic in D if at each U-point, if such exist, the mapping is sense-preserving; here a point $z_0\in D$ is called a U-point of $f(z)$ if there exists a neighbourhood D_{z_0} of z_0 such that the image of an arbitrary point $z\in D_{z_0}$, $z\neq z_0$, is different from $f(z_0)$.

It is also shown that if $f(z)$ is continuous in D with a finite $\rho(z)$ which can vanish on at most a denumerable set in D, then there exists an open set O everywhere dense in D in every component of which either $f(z)$ or $\overline{f(z)}$ is analytic; moreover there exists a set of analytic arcs everywhere dense in $K=D-O$ such that in the vicinity of any one of these arcs $f(z)$ is conformally equivalent to Bohr's function $b(z)$ [$=z$ (Im $z>0$); $=\bar{z}$ (Im $z<0$)]. It is shown that the same result ensues if it be assumed merely that $\rho(z)$ is continuous in D.

A. J. Lohwater (Ann Arbor, Mich.)

120:

Pfluger, Albert. Über die Äquivalenz der geometrischen und der analytischen Definition quasikonformer Abbildungen. Comment. Math. Helv. **33** (1959), 23–33.

An orientation-preserving homeomorphism $\zeta(z)$ of a domain D is called geometrically K-quasiconformal if the module M of a generalised quadrilateral Ω in D and the module M' of $\zeta(\Omega)$ satisfy the inequality $K^{-1}M\leq M'\leq KM$, while the homeomorphism $\zeta(z)$ is called analytically K-quasiconformal if: (1) $\zeta(z)$ is absolutely continuous in D in the sense of Tonelli (AST in D); (2) the partial derivatives ζ_x, ζ_y are locally square integrable; and (3) almost everywhere in D,

$$\max_{(\vartheta)} |\, \zeta_x \cos\vartheta + \zeta_y \sin\vartheta|^2 \leq KJ,$$

where J is the Jacobian of the mapping $\zeta(z)$. The author

gives a new, detailed proof of the following result of Mori [Trans. Amer. Math. Soc. **84** (1957), 56–77; MR **18**, 646]: A homeomorphism $\zeta(z)$ of a domain D is geometrically K-quasiconformal in D if and only if $\zeta(z)$ is AST in D and satisfies (3) almost everywhere in D.

A. J. Lohwater (Ann Arbor, Mich.)

121a:

Bilimović, Anton. Sur la déflexion d'une fonction non-analytique du quaternion par rapport à une fonction analytique. Glas Srpske Akad. Nauka **228** Od. Prirod.-Mat. Nauka (N.S.) **13** (1957), 1–22. (Serbo-Croatian. French summary)

121b:

Bilimovitch, Anton. Sur la déflexion d'une fonction non-analytique du quaternion par rapport à une fonction analytique. Bull. Acad. Serbe Sci. (N.S.) **20** Cl. Sci. Math.-Nat. Sci. Math. **3** (1957), 1–9.

The author has previously shown [same Glas **9** (1956), 1–11; MR **19**, 537] that, for a complex function $f(z)=P(x,y)+iQ(x,y)$, the expression $\bar{B}=\operatorname{grad} Q-\bar{k}\times\operatorname{grad} P$, where $\bar{k}$ is the unit vector normal to the z-plane, vanishes whenever $f(z)$ is analytic, and, in the general case, serves as a measure of how much $f(z)$ deviates from being analytic. In the present papers (the second being a summary of the first), the author extends as far as he can this concept of deviation from analyticity to the case of quaternion functions. He defines and discusses the vector differential calculus of the quaternion functions and, because of the non-commutative nature of quaternion multiplication, the measure of the deflection of a quaternion function is limited to differential quotients along certain directions. The author is forced to the conclusion that there is no meaningful analogue of a monogenic or analytic function in the classical sense, except for the trivial case of the linear quaternion function.

A. J. Lohwater (Ann Arbor, Mich.)

122:

Huan, Le-de. Some fundamental theorems on holomorphic vectors. Sci. Record (N.S.) **2** (1958), 53–58. (Russian)

The author proves generalizations of Liouville's theorem and the maximum modulus theorem for holomorphic vectors $\Phi(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ defined by the differential equations

$$\partial_x\Phi_2+\partial_y\Phi_3+\partial_z\Phi_4=0,\quad \partial_x\Phi_1+\partial_y\Phi_4-\partial_z\Phi_3=0,$$
$$\partial_y\Phi_1+\partial_z\Phi_2-\partial_x\Phi_4=0,\quad \partial_z\Phi_1-\partial_y\Phi_2+\partial_x\Phi_3=0.$$

H. Tornehave (Copenhagen)

123:

Erdös, P.; Herzog, F.; and Piranian, G. Metric properties of polynomials. J. Analyse Math. **6** (1958), 125–148.

In this paper a number of interesting theorems are proved about the point set E on which the polynomial $f(z)=(z-z_1)\cdots(z-z_n)$ has a modulus less than one. The case that all the z_j lie on the interval $[-1, 1]$ of the real axis L is treated first; if the centroid of the z_j lies on $[0, 1]$, it is shown that $E\cap L$ contains the interval $(0, 1)$ and has length at least $\sqrt{2}$ but does not intersect the interval $(-\infty, -\sqrt{2}]$. This result is an improvement on those of Erdös [Bull. Amer. Math. Soc. **46** (1940), 954–958; MR **1**, 242] and Steinberg [Amer. Math. Monthly **59** (1952), 420–421]. In the case that all z_j lie on interval $[-r, r]$ of L, the supremum of the diameter of $E\cap L$ is found to be $2(1+r^2)^{\frac{1}{2}}$ if $0\leqq r\leqq \frac{3}{4}$ and $1+2r$ if $(\frac{3}{4}\leqq r<\infty)$. If all the z_j lie on the unit circle C, it is shown that the inequality $0<|E\cap C|<2\pi$ is valid, with 0 and 2π the best possible such constants. The measure of E is also studied when the z_j lie on the closure of the unit disk or, generally, on a set of transfinite diameter less than one. The number of components in E and the sum of their diameters are also studied under various conditions.

M. Marden (Milwaukee, Wis.)

124:

Wu, Zwao-Jen. Some classes of functions of star-likeness. Acta Math. Sinica **7** (1957), 167–182. (Chinese. English summary)

The author studies classes of star-like and convex functions univalent in the unit circle $|z|<1$, with order ρ of star-likeness and convexity in the sense of M. S. Robertson [Ann. Math. **37** (1936), 374–408] and with p-fold symmetry of rotation. He considers: (a) the class $S_p^*(\rho)$ of functions

$$f(z)=z+\sum_{\nu=1}^{\infty}a_{\nu p+1}^{(p)}z^{\nu p+1}$$

regular and univalent in the unit circle and such that

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\}\geqq\rho,\qquad 0\leqq\rho<1,\quad |z|<1;$$

(b) the class $K(\rho)$ of functions

$$g(z)=z+\sum_{\nu=1}^{\infty}b_{\nu p+1}^{(p)}z^{\nu p+1}$$

regular and univalent in the unit circle and such that

$$\operatorname{Re}\left\{1+z\frac{g''(z)}{g'(z)}\right\}\geqq\rho,\qquad 0\leqq\rho<1,\quad |z|<1.$$

If $g(z)\in K(\rho)$, then $zg'(z)\in S_p^*(\rho)$. Associated with $S_p^*(\rho)$ is the class of functions Σ_p^*

$$F(\zeta)=\zeta\left(1+\sum_{\nu=1}^{\infty}a_{\nu p}^{(p)}\zeta^{-\nu p}\right)$$

mapping the region $|\zeta|>1$ into the plane with p-fold symmetric complement of order ρ of star-likeness, where $1/F(\zeta)=f(z)\in S_p^*(\rho)$ with $z=1/\zeta$. Structure formula are established for functions of the classes $S_p^*(\rho)$, $K(\rho)$, Σ_p^*.

As a necessary and sufficient condition for a function to belong to a given one of the above-mentioned classes, it is established that the function must be representable in the form:

$$f(z)=z\exp\left\{-\frac{1-\rho}{p\pi}\int_0^{2\pi}\ln\,(1-z^pe^{-i\theta})d\alpha(\theta)\right\}$$

for the class $S_p^*(\rho)$;

$$g(z)=\int_0^z\exp\left\{-\frac{1-\rho}{p\pi}\int_0^{2\pi}\ln\,(1-z^pe^{-i\theta})d\alpha(\theta)\right\}dz$$

for the class $K(\rho)$;

$$F(\zeta)=\zeta\exp\left\{\frac{1-\rho}{p\pi}\int_0^{2\pi}\ln\left(1-\frac{e^{-i\theta}}{\zeta^p}\right)d\alpha(\theta)\right\}$$

for the class Σ_p^*, where $0\leqq\rho<1$, $p=1, 2, \cdots$, $\alpha(\theta)$ is a real

non-decreasing function on $[0, 2\pi]$ normed by the condition $\int_0^{2\pi} d\alpha(\theta) = 2\pi$, the symbol ln denotes the principal branch and the integrals are taken in the sense of Stieltjes.

With the help of the structure formulas set up, a series of extremal problems is solved and univalent majorants are constructed.

In particular, estimates are obtained for the absolute values

$$\left|1-\left[\frac{f(z)}{z}\right]^{-\lambda/2}\left[\frac{2}{3}\left(z\frac{f'(z)}{f(z)}\right)+\frac{1}{2}\right]^{\lambda-1}\right|, \quad 0 \leq \lambda \leq 1,$$

$$\left|\arg\frac{f(z)}{z}\right|, \qquad \left|\arg z\frac{f'(z)}{f(z)}\right|, \qquad |Af(z)+Bzf'(z)|,$$

for the limit of convexity of order p for the class of univalent functions with p-fold symmetry of rotation, etc. There is a bibliography with 11 entries.

S. A. Kas'yanyuk (RŽMat **1958** #6630)

125:

Derwidué, L. Sur le nombre des racines à partie réelle positive des équations algébriques. Mathesis **66** (1957), 144–151.

Bückner [Quart. Appl. Math. **10** (1952), 205–213; MR **14**, 145] gave a determinantal representation for a Hurwitz polynomial. Using the same presentation for the real polynomial $f(z) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + 1$, Derwidué shows that $f(z)$ has as many zeros with positive real part as the number of negative coefficients in the presentation. This result may be expressed in terms of the Hurwitz determinants of (z). [Cf. Marden, *Geometry of the zeros of a polynomial in a complex variable*, Amer. Math. Soc., New York, 1949; MR **11**, 101; p. 141].

M. Marden (Milwaukee, Wis.)

126:

Parodi, Maurice. Sur la localisation des zéros des polynomes lacunaires. Bull. Sci. Math. (2) **82** (1958), 67–72.

The author writes the polynomial

$$f(z) = z^n + a_pz^{n-p} + \cdots + a_{n-1}z + a_n$$

in the form

$$f(z) = \begin{vmatrix} -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -z & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_{p+1} & -a_p - z^p \end{vmatrix}$$

Using the Hadamard condition for the non-vanishing of a determinant, the author concludes that all the zeros of $f(z)$ lie in the union of the interiors of the curves

$$|z| = 1 \quad \text{and} \quad |z^p + a_p| = |a_{p+1}| + \cdots + |a_n|.$$

He applies this result to the trinomial and quadrinomial equations.

M. Marden (Milwaukee, Wis.)

127:

Parodi, Maurice. Complément à un travail sur les polynomes lacunaires. C. R. Acad. Sci. Paris **247** (1958), 908–910.

The author shows that polynomial

$$f(z) = z^n + a_pz^{n-p} + \cdots + a_{n-1}z + a_n$$

may be written in the form of a determinant to which Hadamard's condition may be applied. The zeros of $f(z)$ are thus found to lie in the union of the regions interior to the curves

$$|z| = n,$$

$$|z^p + a_p| = \frac{|a_n|}{(p+1)\cdots n} + \frac{|a_{n-1}|}{(p+1)\cdots(n-1)} + \cdots + \frac{|a_{p+1}|}{p+1}.$$

M. Marden (Milwaukee, Wis.)

128:

Trohimčuk, Yu. Yu. Sur la généralisation de théorème de Picard. Ukrain. Mat. Ž. **10** (1958), no. 1, 70–77. (Russian. French summary)

A domain G in the z-plane is a Liouville domain provided every function holomorphic and bounded in G is constant in G; a point set is a Liouville set if it is the boundary of a Liouville domain. Fundamental lemma: If Γ is a nonempty Liouville set contained in a domain D, and if f is holomorphic in $D\backslash\Gamma$ ($\equiv D-\Gamma$) and has an essential singularity at each point of Γ, then the set $f(D\backslash\Gamma)$ is a Liouville domain. Extension of Picard's theorem: If f is meromorphic in a domain D, except for a nonempty set Γ of essential singularities, and if Γ is a Liouville set, then the set of values w that f takes only finitely often in D is a set of measure zero and first category.

G. Piranian (Ann Arbor, Mich.)

129:

Trohimčuk, Yu. Yu. On conformal mappings. Dokl. Akad. Nauk SSSR **121** (1958), 430–431. (Russian)

The author generalizes theorems of Menshov and Bohr concerning analyticity of complex-valued functions f defined in a domain D of the z-plane. Example: If f is univalent and continuous in D, and if the (finite or infinite) coefficient of dilation $\lim_{h\to 0}|h^{-1}[f(z+h)-f(z)]|$ exists everywhere in D, except possibly at a denumerable point set, then either f or the conjugate of f is analytic in D.

G. Piranian (Ann Arbor, Mich.)

130:

Chu, Soon-hu. Functions with images of bounded measure. Advancement in Math. **2** (1956), 667–674. (Chinese. English summary)

This note is concerned with the class S_M of functions $f(z) = a_0 + a_1z + \cdots$, $|z| < 1$, such that the relation $w = f(z)$ maps $|z| < 1$ in $D(f)$ on the w-plane, with the measure less than or equal to M. The surface $D(f)$ may be of several sheets, the multiplicity is to be taken according to the number of sheets. Functions of S_M satisfying the condition $f'(0) \neq 0$ form a sub-class S_M'. We denote by S_M'' the sub-class of S_M' of which each $f(z)$ is schlicht in the unit circle.

For the function $f(z)$ of S_π the coefficients satisfy the inequality $\sqrt{n}|a_n| \leq 1$ $(n = 1, 2, \cdots)$, and the equality $\sqrt{n}|a_n| = 1$ holds when and only when $f(z) = a_0 + \varepsilon n^{-\frac{1}{2}}z^n$, $|\varepsilon| = 1$. The distortion theorem for $f(z) \in S_\pi$ states that

$$(1) \qquad |f'(z)| \leq \frac{1}{1-r^2}, \quad |z| = r < 1,$$

where the sign of equality holds only for the function

$$f(z) = a_0 + \varepsilon\frac{\alpha + z}{1+\bar{\alpha}z}, \quad |\varepsilon| = 1$$

with some proper α, $|\alpha|<1$. It follows from (1) that

$$|f(z)-f(0)| < \log \frac{1+r}{1-r}.$$

This estimation is, however, not precise. We leave here the question of precision open.

Let the circle $|z|=\rho$, $\rho<1$, be represented by the function $w=f(z)\in S_\pi$, and denote the length of the image curve by $C(\rho)$. We have the following theorem: If $0<\rho\leqq\frac{1}{2}$, then $C(\rho)\leqq 2\pi\rho$, and if $\frac{1}{2}<\rho<1$, then

$$C(\rho) \leqq \tfrac{1}{2}\pi/(1-\rho). \tag{2}$$

The inequality $C(\rho)\geqq 2\pi|f'(0)|$ holds true for $0<\rho<1$. These estimates, except (2), are precise, the extremal function is $f(z)=a_0+\varepsilon z$, $|\varepsilon|=1$.

As for $f(z)\in S_\pi''$, we prove the following two theorems: (a) if p and q are positive integers such that $p<q$, then the function $[f(z^q)-f(0)]^{p/q}$ belongs to $S_{p\pi}$; (b) if $0\leqq\alpha\leqq 1$, then the coefficients of the expansion

$$\left(\frac{f(z)-f(0)}{z}\right)^\alpha = \sum_{n=0}^{\infty} B_n(\alpha)z^n$$

satisfy the inequality

$$\sum_{n=0}^{\infty} (n+\alpha)\cdot|B_n(\alpha)|^2 \leqq \alpha.$$

Author's summary

131:

Kuroda, Tadashi. Remarks on some covering surfaces. Rev. Math. Pures Appl. **2** (1957), 239–244.

Soit $w=f(p)$ une transformation intérieure définie sur une variété V à deux dimensions et à valeurs dans le plan complexe Σ. L'auteur introduit la surface de recouvrement $\Phi=(\Sigma)_f^v$ (notation de Stoïlov), definit une exhaustion de V. Soit E un ensemble fermé $E\subset\Sigma$, dont le complémentaire D soit un domaine. Si D est exhaustible moyennant certaines conditions sur les frontières des domaines de la suite d'exhaustion, l'auteur définit une exhaustion E-quasi-normale de Φ. L'auteur démontre alors quelques propriétés sur les surfaces Φ, dont les plus intéressantes sont les propriétés de Gross, lorsque E est de mesure linéaire nulle, et d'Iversen lorsque E est totalement discontinu.

Il est à remarquer toutefois que l'introduction de $\Phi=(\Sigma)_f^v$ ne s'impose pas puisque f est uniforme et que les propriétés obtenues sur Φ sont en fait obtenues sur V projetée par f sur Σ. *L. Fourès* (Marseille)

132:

Sainouchi, Yoshikazu. A remark on the polygonal representation of Riemann surfaces. Bull. Univ. Osaka Prefecture Ser. A **6** (1958), 1–9.

Doubly connected Riemann surfaces R can be constructed from the half-strip $x>0$, $0\leqq y\leqq 1$ by identifying the boundary half-lines in various manners. The resulting surface is called parabolic or hyperbolic according as its modulus is infinite or finite. The author generalizes the parabolicity tests of R. Nevanlinna [Ann. Acad. Sci. Fenn. Ser. A. I, no. **122** (1952); MR **14**, 743] and of L. I. Volkovyskiĭ [Mat. Sb. (N.S.) **18** (**60**) (1946), 185–212; MR **8**, 326] to the following form. R is parabolic if $(x,0)$ is identified with $(x+f(x),1)$ by a positive analytic function $f(x)$ subject to these conditions: (1) $\liminf_{x\to\infty} f(x)>0$ and $1+f'(x)>\text{const.}>0$; (2) there exists a sequence $\{a_n\}$, $a_n\to\infty$, such that $a_{n+1}\geqq a_n+f(a_n)$; (3) $\sum_{n=0}^{\infty} 1/M_n=\infty$, where $M_n=\max f(x)$ in $a_n\leqq x\leqq a_{n+1}$. An extension is also given to the type criterion of C. Blanc [Comment. Math. Helv. **11** (1939), 130–150], and related results are derived for double strips. *L. Sario* (Los Angeles, Calif.)

133:

Kneser, Hellmuth. Analytische Struktur und Abzählbarkeit. Ann. Acad Sci. Fenn. Ser. A. I. no. **251/5** (1958), 8 pp.

The starting point of this investigation is the well-known theorems of T. Radó [Acta Sci. Math. Szeged **2** (1925), 101–121] and E. Calabi and M. Rosenlicht [Proc. Amer. Math. Soc. **4** (1953), 335–340; MR **15**, 351]. The former states that every 2-dimensional manifold with conformal structure is countable; the latter, that there exist noncountable complex manifolds of higher dimension. The present author raises the question of the existence of other structures that do not imply countability for all dimensions. He establishes the following result: for every $n>0$ there are (connected) noncountable n-dimensional manifolds with real analytic structures. For $n>1$ the theorem is a consequence of the Calabi-Rosenlicht counterexample [loc. cit.]. For $n=1$ the proof is based on an example of Alexandroff [Math. Ann. **92** (1924), 294–301].

It is shown, moreover, that every 1-dimensional manifold can be endowed with a real analytic structure. *L. Sario* (Los Angeles, Calif.)

134:

Rosenbloom, P. C. The inequalities of Ehrenpreis, Malgrange and Hörmander. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/31** (1958), 15 pp.

Let $f(z_1,\cdots,z_n)$ be an entire analytic function of exponential type and $|f|_p$ its L^p-norm for real arguments. The inequality $|f|_p\leqq A|Pf|_p$, where $P\neq 0$ is a polynomial and A is a constant depending on P and the exponential type of f, have been used by Malgrange [Ann. Inst. Fourier Grenoble **6** (1955/56), 271–355; MR **19**, 280], Ehrenpreis [Amer. J. Math. **76** (1957), 883–903; MR **16**, 834] and Hörmander [Acta Math. **94** (1955), 161–248; MR **17**, 853] in connection with the construction of elementary solutions of the differential operator with characteristic polynomial P.

The author gives two new proofs and estimates of A and constructs an elementary solution which does not grow too fast at infinity. *L. Gårding* (Lund)

135:

Ostrovskiĭ, I. V. On meromorphic functions taking certain values at points lying near a finite system of rays. Dokl. Akad. Nauk SSSR **120** (1958), 970–972. (Russian)

The paper contains statements of results and brief indications of the method of proof. For the sake of brevity the theorems are not quoted here in full generality.

Let $f(z)$ be a mermorphic function with poles $r_ke^{i\varphi_k}$. Write $C(R,\alpha,\beta,f)=2\sum(r_k^{-\pi/\delta}-r_k^{\pi/\delta}R^{-2\pi/\delta})\sin\pi(\varphi_k-\alpha)/\delta$, when the summation is over all poles with $1\leqq r_k<R$, $\alpha<\varphi_k<\beta=\alpha+\delta$. Denote the Nevanlinna characteristic function of $f(z)$ by $T(r)$. The a-points of $f(z)$ are called close to the rays (1) $\arg z=\theta_k$, $k=1,2,\cdots,n$, $0\leqq\theta_1<\theta_2\cdots<\theta_n<\theta_{n+1}=\theta_1+2\pi$, if $\sum_{k=1}^{n} C(R,\theta_k,\theta_{k+1},1/(f-a))=O(T^\kappa(R))$, $\kappa<1$.

Write γ for min $(\theta_{k+1}-\theta_k)$. Theorem 1. The function $f(z)$ is at most of order $\pi/[\gamma(1-k)]$, if one of the hypotheses A–C holds: (A) The poles and zeros of $f(z)$ are close to (1), and at least one derivative of $f(z)$ has a deficient value c distinct from 0 and ∞; (B) the poles of $f(z)$ and the c-points of a derivative $f^{(l)}(z)$ are close to (1), where $c\neq 0$, ∞, and 0 is a deficient value of $f(z)$; (C) the zeros and poles of $f(z)$ and the c-points of $f^{(l)}(z)$ are close to (1) ($c\neq 0$, ∞), and ∞ is a deficient value of $f(z)$.

For functions meromorphic in $|z|<1$ similar results are obtained. We quote theorem 2. If $f(z)$ is meromorphic in $|z|<1$ and if the zeros and poles of $f(z)$ and the c-points of a derivative of $f(z)$ ($c\neq 0$, ∞) lie on a finite number of rays, then $\limsup \log T(r)/\log[1/(1-r)]\leq 4$.

The basic tool in the proofs is an estimate of

$$\int_\alpha^\beta \log^+|f^{(l)}(Re^{i\theta})/f^{(l+1)}(Re^{i\theta})|\,d\theta$$

in terms of the expressions $C(R',\alpha,\beta,f)$, $C(R',\alpha,\beta,1/f)$ $(R'>R)$.

Theorem 1 generalizes work by A. Edrei [Trans. Amer. Math. Soc. **78** (1955), 276–293; MR **16**, 808].

W. H. J. Fuchs (Ithaca, N.Y.)

136:

Gabib-Zade, A. Š. Investigation of a non-linear Hilbert problem. Dokl. Akad. Nauk Azerbaĭdžan. SSR **14** (1958), 275–278. (Russian. Azerbaijani summary)

Given a bounded multiply-connected region σ^+ bounded by contours Γ, let σ^- denote the complement of $\sigma^+\cup\Gamma$, and let Φ^+, Φ^- denote the boundary values of Φ from inside σ^+ and σ^-, respectively. The author sets the problem: to find a piecewise meromorphic solution of

$$\Phi^+(t)-A(t)\Phi^-(t)-B(t)\Phi^+(t)\Phi^-(t) = C(t)$$

with given A, B, C satisfying Hölder conditions on Γ. Let

$$2\pi i\kappa = \{\log[A(t)+B(t)C(t)]\}_\Gamma.$$

The author finds that if $\kappa\geq 0$ then there is an at least κ-parameter family of solutions, for which he obtains a formula depending on the solution of a singular integral equation. *R. P. Boas, Jr.* (Evanston, Ill.)

137:

Davydov, N. A. Radial boundary values of analytic functions. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 157–164. (Russian)

If a series $f(z)=\sum c_nP_n(z)$, where $P_n(z)$ is a polynomial of degree n, converges uniformly in K ($|z|<1$), then $F_R(z)=\sum (c_n/R^n)P_n(z)$ converges uniformly in K_R ($|z|<R$), where $R>1$. For the special case that $P_n(z)=z^n$, it is known that $G\subset G_R$ and that $G_R\to G$ as $R\to 1$, where G and G_R denote the stars of $f(z)$ and $F_R(z)$, respectively. The author studies the extent to which these properties may be carried over to the case of the general polynomial $P_n(z)$. Examples are constructed giving negative answers to these questions, and a further example shows that even if the series representing $f(z)$ in K has radial (or angular) boundary values on a set E on $|z|=1$, the corresponding function $F_R(z)$ need not have radial (or angular) boundary values at any point of the radial projection of E onto $|z|=R$. *A. J. Lohwater* (Ann Arbor, Mich.)

138:

Liu, Li-Chuan. Some inequalities derived from fundamental lemma concerning schlicht functions. Acta Math. Sinica **7** (1957), 313–326. (Chinese. English summary)

Let S be the class of functions

$$f(z) = z+a_2z^2+a_3z^3+\cdots,$$

regular and univalent in the circle $|z|<1$; let S_M be the subset of S with $|f(z)|<M$ for $|z|<1$; let Σ be the class of functions

$$F(\zeta) = \zeta+\alpha_0+\frac{\alpha_1}{\zeta}+\frac{\alpha_2}{\zeta^2}+\cdots,$$

regular and univalent in the region $1<|\zeta|<\infty$; let Σ_R be the subset of Σ with $|F(\zeta)|>R$ for $|\zeta|>1$.

The author proves the following. Theorem 1: If $F(\zeta)\in\Sigma$, then

$$|\alpha_5+\alpha_1\alpha_3+\alpha_2^2+\tfrac{1}{3}\alpha_1^3| \leq \tfrac{1}{3},$$

with equality only for $\alpha_3=0$, $\alpha_1\alpha_2+\alpha_4=0$. In the case $\alpha_1=0$, the equality holds if and only if

$$F(\zeta) = \zeta\left(1-\frac{\eta x}{\zeta^3}+\frac{\eta^2}{\zeta^6}\right)^{\frac{1}{3}}+\text{const.},\quad 0\leq|x|\leq 2,\quad |\eta|=1.$$

Theorem 2: If $F(\zeta)\in\Sigma$, then $|\alpha_1\alpha_2|<P$, $P=0.311\cdots$ where

$$P=\frac{4}{9}x_0+\frac{2}{27}x_0^2-\frac{4}{27}x_0^3-\frac{14}{243}x_0^4-\frac{4}{729}x_0^5,$$

and $x_0=0.92402\cdots$ is the positive root of the equation $5x^3+27x^2-27=0$. The equality holds only for

$$F(\zeta) = \zeta\left(1+\frac{x_0\eta}{\zeta}-\frac{x_0\eta^2}{\zeta^2}-\frac{\eta^3}{\zeta^3}\right)^{\frac{1}{3}}+\text{const.},\quad |\eta|=1.$$

Theorem 3: If $f(z)=z+a_{n+1}(M)z^{n+1}+\cdots$ is in S_M, then the sharp estimate holds:

$$|a_{\mu+1}(M)|\leq\frac{2}{\mu}\left(1-\frac{1}{M^\mu}\right),\quad \mu=n,\cdots,2n.$$

Theorem 4: If $F(\zeta)=\zeta+\alpha_0+\alpha_n(R)/\zeta^n+\cdots$ is in Σ_R, then the sharp estimate holds:

$$|\alpha_{\mu-1}(R)|\leq\frac{2}{\mu}(1-R^\mu),\quad \mu=n+1,\cdots,2n+1.$$

For $M\to\infty$ and $R\to 0$ theorems 3 and 4 reduce to Goluzin's theorems. *V. N. Teliyanc* (RŽMat **1959** #3702)

139:

Gal'perin, I. M. Finite sections of the Taylor series of two special classes of univalent functions. Uspehi Mat. Nauk **13** (1958), no. 5 (83), 171–178. (Russian)

$U_{(0)}$ is the class $\{\varphi(z)\}$ of analytic functions defined by

$$(*)\qquad \varphi(z) = z+\sum_{n=2}^\infty a_nz^n = -(1/2\pi)\int_0^{2\pi}[z+2e^{-it}\ln(1-ze^{-it})]d\mu(t);$$

and $K+S$ the class $\{\Phi(z)\}$ given by

$$(**)\quad \Phi(z) = z+\sum_{n=2}^\infty b_nz^n = (1/2\pi)\int_0^z\left\{\int_0^{2\pi}\frac{d\mu(t)}{1-\zeta e^{-it}}\times\exp\left[-(1/\pi)\int_0^{2\pi}\ln(1-\zeta e^{-it})d\mu(t)\right]\right\}d\zeta.$$

In both cases, $\mu(t)$ is non-decreasing on $[0, 2\pi]$, with $\int_0^{2\pi} d\mu(t) = 2\pi$. The following results are proved. (i) Let $\varphi \in U_{(0)}$ be univalent in $|z|<1$. For all $n \geqq 2$ the power sections $s_n(z) = z + \sum_{k=2}^n a_k z^k \in U_{(0)}$ and are univalent in $|z| < \frac{1}{2}$, and the constant $\frac{1}{2}$ cannot be increased. (ii) If $\Phi(z) \in K+S$, then for all $n \geqq 2$ its sections $\sigma_n(z) = z + \sum_{k=2}^n b_k z^k$ are univalent in $|z| < \frac{1}{3}$, the constant $\frac{1}{3}$ cannot be increased. *I. M. Sheffer* (University Park, Pa.)

140:

Liu, Li-chuan. Bounded schlicht functions in the unit circle. Acta Math. Sinica 7 (1957), 439–450. (Chinese. English summary)

Let B_k $(k=1, 2, \cdots)$ be the class of functions

$$f(z) = az + \sum_{n=1}^{\infty} a_{nk+1} z^{nk+1}, \quad |a| < 1,$$

which are regular and univalent in $|z|<1$ and such that $|f_k(z)|<1$ for $|z|<1$. With the help of Loewner's equation the author proves the following theorem. Let $f_k(z)$ belong to B_k. We set $|z|=r$, $\rho = \{2k-(3k^2+1)^{\frac{1}{2}}\}^{1/k}(k-1)^{-1/k}$, and let λ denote the smallest positive root of the equation

$$\frac{k+1+(k-1)\lambda^{2k}}{k(1+\lambda^k)^2} = \frac{4r^k}{(1+r^k)^2}.$$

With these notations we have: (1) if

$$|a| \leqq \frac{\lambda}{r}\left(\frac{1-r^k}{1-\lambda^k}\right)^{2/k},$$

then

$$|f'_k(z)| \leqq \frac{R}{r}\frac{1+r^k}{1-r^k}\frac{1-R^k}{1+R^k},$$

where R is the positive root of the equation $R^k/(1-R^k)^2 = |a|^k r^k/(1-r^k)^2$; (2) if

$$\frac{\lambda}{r}\left(\frac{1-r^k}{1-\lambda^k}\right)^{2/k} < |a| < \left(\frac{\rho}{r}\right)^{(1+\rho^k)/(1-\rho^k)},$$

then

$$|f'_k(z)| \leqq \frac{r}{x}\frac{1-R^k}{1-r^k}\frac{1+x^k}{1+r^k}|a|,$$

where the positive numbers R and x are determined from the equations

$$\frac{R}{x}\frac{1+x^k}{1+R^k}\left(\frac{x}{r}\right)^{(1-x^k)/(1+x^k)} = |a|,$$

$$\frac{k+1+(k-1)R^{2k}}{k(1+R^k)^2} = \frac{4x^k}{(1+x^k)^2};$$

(3) if $|a| \geqq (\rho/r)^{(1+\rho^k)/(1-\rho^k)}$, then

$$|f'_k(z)| \leqq \frac{1-R^{2k}}{1-r^{2k}}\left(\frac{r}{R}\right)^{(k-1-(3k-1)R^{2k})/2(1-R^{2k})},$$

where R is the real root of the equation

$$\ln\frac{1}{|a|} = 2k\left(k-1+\frac{2kR^{2k}}{1-R^{2k}}\right)\ln\frac{r}{R}$$

The estimates (1), (2), (3) are sharp.

{Reviewer's remarks. (1) There are many misprints. (2) The second equation in (3), regarded as an equation in R, may have two positive roots. The author does not indicate which of the two roots should be chosen in this case (obviously the smaller). (3) Let us denote by $S_M^{(1)}$ the class of functions $f(z) = z + \sum_{n=2}^{\infty} c_n z^n$ regular and univalent in the unit circle $|z|<1$ and such that $|f(z)|<M$, $M>1$ in $|z|<1$. If $f_k(z) \in B_k$, then

$$f(z) = [f_k(z^{1/k})/a]^k \in S_{1/|a|}^{(1)}.$$

Conversely, if $f(z) \in S_M^{(1)}$, then $f_k(z) = \{f(z^k)/M\}^{1/k} \in B_k$, with $a = 1/M$.

It is of interest to rephrase the theorem for the class $S_M^{(1)}$ by making use of this relation. We then get an estimate from above for the magnitude $|z^{(k-1)/k}f'(z)/f(z)^{(k-1)/k}|$, $f(z) \in S_M^{(1)}$ where k is a natural number. The question to be cleared up is: for what real k is the estimate obtained in this way correct. See also the article by Sung-shih Tang reviewed below. *N. A. Lebedev* (RŽMat **1958** #7643)

141:

Tang, Sung-shih. On the distortions of a class of schlicht functions. Advancement in Math. 3 (1957), 478–484. (Chinese. English summary)

Let $\Sigma_R^{(k)}$ be the class of functions

$$F_k(\zeta) = \zeta + \sum_{n=1}^{\infty}\frac{a_n^{(k)}}{\zeta^{nk-1}} \quad (k = 1, 2, \cdots)$$

regular and univalent in $1<|\zeta|<\infty$ and satisfying there the condition $|F_k(\zeta)|>R$, $(R \geqq 0)$.

The author, on the basis of Loewner's equation, sharpens and generalizes various known estimates (G. M. Goluzin, I. E. Bazilevič, N. A. Lebedev, I. M. Milin, and others) of the type of distortion theorems in classes $\Sigma_R^{(k)}$. For example, the following is obtained: let ζ_ν, ζ_ν' $(\nu = 1, 2, \cdots, n)$ be complex numbers from $|\zeta|>1$, let γ_ν, γ' be arbitrary complex numbers, and let $a_{\nu,\mu}$ $(a_{\nu,\mu} = a_{\mu,\nu})$ be real numbers such that, for all real x_ν, $\sum_{\nu,\mu=1}^n a_{\nu\mu}x_\nu x_\mu \geqq 0$; if $F_l(\zeta) \in \Sigma_R^{(1)}$, $l = 1, 2, \cdots, m$, then

$$\left\{\left|\sum_{\nu,\mu=1}^n a_{\nu,\mu}\gamma_\nu\gamma_\mu' \prod_{j=1}^m \ln Q_{F_j}(\zeta_\nu, \zeta_\mu')\right|\right\}^2 \leqq$$

$$\sum_{\nu,\mu=1}^n a_{\nu,\mu}\gamma_\nu\bar{\gamma}_m \prod_{j=1}^m \ln\frac{1-\dfrac{R^2}{F_j(\zeta_\nu)\overline{F_j(\zeta_\mu)}}}{1-\dfrac{1}{\zeta_\nu\bar{\zeta}_\mu}}$$

$$\times\sum_{\nu,\mu=1}^n a_{\nu,\mu}\gamma_\nu'\bar{\gamma}_\mu' \prod_{j=1}^m \ln\frac{1-\dfrac{R^2}{F_j(\zeta_\nu')\overline{F_j(\zeta_\mu')}}}{1-\dfrac{1}{\zeta_\nu'\bar{\zeta}_\mu'}},$$

where $Q_F(\zeta, \zeta') = (F(\zeta)-F(\zeta'))/(\zeta-\zeta')$ if $\zeta \neq \zeta'$ and $Q_F(\zeta, \zeta') = F'(\zeta)$ if $\zeta = \zeta'$.

Let us denote by S_M' the class of functions $f(z) = (F(1/z))^{-1}$, where $F(\zeta) \in \Sigma_{1/M}^{(1)}$ and $F(\zeta) \neq 0$ for $1<|\zeta|<+\infty$. Then $(|z|=r,\ |f(z)|=f)$:

$$\left|\ln\frac{z^\lambda f'(z)^{1-\lambda}}{f(z)^\lambda}\right| \leqq \begin{cases} \ln\left[\left(\dfrac{1+r}{1+f/M}\right)^{1-\lambda}\left(\dfrac{1-f/M}{1-r}\right)^{3-5\lambda}\right], & \lambda \leqq \frac{1}{2}; \\ \ln\left[\left(\dfrac{1+r}{1+f/M}\right)^{3-5\lambda}\left(\dfrac{1-f/M}{1-r}\right)^{1-\lambda}\right], & \frac{1}{2} \leqq \lambda \leqq \frac{2}{3}; \\ \ln\left[\left(\dfrac{1+r}{1+f/M}\right)^{\lambda-1}\left(\dfrac{1-f/M}{1-r}\right)^{5\lambda-3}\right], & \lambda \geqq \frac{2}{3}. \end{cases}$$

The right side of the inequality depends on $|f(z)|$, differently for different λ.

{Reviewer's remark: It would be interesting to obtain an estimate for the magnitude in question such that the right side would depend only on $|z|$ [cf. N. A. Lebedev, Vestnik Leningrad. Univ. **10** (1955), no. 8, 29–41; MR **17**, 248].} *N. A. Lebedev* (RŽMat **1958** #7642)

142:

Krzyż, J. On the derivative of bounded univalent functions. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 157–159.

The author gives an elementary derivation of the precise lower bound for $|f'(z)|$ where $f(z)=z+a_2z^2+\cdots$ is regular univalent and bounded ($\sup|f(z)|\leq M$) in the unit disk. Using essentially Pick's [Wien. Berichte **126** (1917), 247–263] method he obtains $|f'|\geq M\phi(|z|)/\phi(|f|/M)$ where $\phi(\alpha)=(1-\alpha)\alpha^{-1}(1+\alpha)^{-1}$. This result which is best possible was obtained by Janowski [Łódzkie Towarzystwo Nauk. Wydział. III **51** (1957)] using the variational method of Charzynski.

The author announces without proof two theorems about the order of $f'(z)$; one of these is: if the function $f(z)$ is regular in the unit disk and if the Riemann surface being the map of this disk is of finite area, then $f'(z)=o(1-|z|)^{-1}$. *V. Linis* (Ottawa, Ont.)

143:

Azpeitia, A. G. On sequences of complex terms defined by iteration. Boll. Un. Mat. Ital. (3) **13** (1958), 522–524. (Italian summary)

Making use of the lemma and theorem II of his paper in Proc. Amer. Math. Soc. **9** (1958), 428–432 [MR **20** #2459] and of a theorem of continuous convergence in a complex space of p dimensions, the author generalizes his previous theorem I (loc. cit., p. 429) by proving that every complex sequence defined by $a_n=f_n(a_{n-p}, a_{n-p+1}, \cdots, a_{n-1})$, with initial terms $a_1, a_2, \cdots, a_p$ arbitrarily chosen in a convex region D of the complex plane, converges if for $z_i\in D$ ($i=1, 2, \cdots, p$) the sequence of functions $f_n(z_1, z_2, \cdots, z_p)$ converges uniformly to a continuous function $f_0(z_1, z_2, \cdots, z_p)$ and if for any $n\geq 0$ the point $w_n=f_n(z_1, z_2, \cdots, z_p)$ always belongs to the closed convex hull R_n of the set of points $z_i\in D$ ($i=1, 2, \cdots, p$) and w_n is different from the extreme points of R_n unless $z_1=z_2=\cdots=z_p$ (then $w_n=z_1$). There seems to be a possibility of other generalizations where there are different infinite partial sequences of functions $f_{n_k}(x_1, x_2, \cdots, x_p)=g_k(x_1, x_2, \cdots, x_p)$ ($k=1, 2, \cdots, m$) having all equal members. *J. Aczél* (Debrecen)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 222.

144:

Temlyakov, A. A. Integral representation of functions of two complex variables. Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 3–9. (Russian)

A domain D in the Riemannian manifold of $w^{1/p}z^{1/q}$, where p and q are relative primes, is bounded by the hypersurface $|w|=r_1(\tau)$, $|z|=r_2(\tau)$, $0<\tau<1$, where $\lim_{\tau\to 0} r_1(\tau)=0$, $0<r_1'(\tau)<p\tau^{-1}r_1(\tau)$, $r_1(1)<\infty$, $r_2(\tau)=\exp(-p^{-1}q\int_0^\tau(1-\tau)^{-1}\tau d\ln r_1(\tau))$. The function $F(w, z)$ is analytic in D and, together with its first order partial derivatives, continuous in $\bar{D}$. The author proves the integral formula

$$F(w,z) =$$

$$(4\pi^2 i)^{-1}\int_0^{2\pi}dt\int_0^1 d\tau\int_{|\xi|=1}(\xi-u)^{-1}\Phi(r_1(\tau)\xi^p,\ r_2(\tau)(\xi e^{-it})^q)d\xi,$$

where

$$\Phi = F+pwF'_w-qzF'_z,$$

$$u = \tau(r_1(\tau))^{-1/p}w^{1/p}+(1-\tau)(r_2(\tau))^{-1/q}z^{1/q}e^{it}.$$

H. Tornehave (Copenhagen)

145:

Temlyakov, A. A. Integral representations of functions of two complex variables. Dokl. Akad. Nauk SSSR **124** (1959) 38–41. (Russian)

Temlyakov's integral representation [Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 89–92; MR **19**, 25] is generalized to domains bounded by hypersurfaces of the type

$$w = r_1(\tau)\zeta,\ z = r_2(\tau)\eta,\ \eta = \zeta e^{-it},\ 0\leq t\leq 2\pi,\ 0<\tau<1,$$

$$r_1(0) = (0),\quad r_1(1)<\infty,\quad r_1'(\tau)>0,\quad \frac{d}{d\tau}(\tau^{-1}r_1(\tau))<0,$$

$$r_2(\tau) = \exp\left(-\int_0^\tau \tau(1-\tau)^{-1}d\ln r_1(\tau)\right),\quad \zeta = \rho(\varphi)e^{i\varphi},$$

where $\rho(\varphi)$ is positive and continuous with period 2π. The generalized formula is

$$F(w,z) =$$

$$(4\pi^2 i)^{-1}\int_0^{2\pi}dt\int_0^1 d\tau\int_{|\zeta|=\rho(\varphi)}(\zeta-u)^{-1}\Phi(r_1(\tau)\zeta,\ r_2(\tau)\eta)d\zeta;$$

$$\Phi = F+wF_w'+zF_z',$$

$$u = \tau(r_1(\tau))^{-1}w+(1-\tau)(r_2(\tau))^{-1}ze^{it}.$$

H. Tornehave (Copenhagen)

146:

Hua, L. K.; and Look, K. H. On Cauchy formula for the space of skew symmetric matrices of odd order. Sci. Record (N.S.) **2** (1958), 19–22.

In the space of m complex variables let $R_{\mathrm{III}}(=R)$ be the domain consisting of all skew-symmetric $n\times n$ matrices Z such that $I+Z\bar{Z}$ is positive definite. For n odd the authors determine explicitly a Cauchy kernel for analytic functions on R, and a Poisson kernel. *F. D. Quigley* (New Orleans, La.)

147:

Hua, L. K.; and Look, K. H. Boundary properties of the Poisson integral of Lie sphere. Sci. Record (N.S.) **2** (1958), 77–80.

Let z be a vector in the space of n complex variables, and let z' be its transpose. The Lie sphere R_{IV} ($=R$) consists of all z such that $1+|zz'|^2-2z\bar{z}>0$ and $1-|zz'|>0$. Let L be the characteristic manifold of R, and B be its boundary. Let G be the stability group of R, and let G' be the subgroup of G of matrices diag$\{1, 1, M\}$ where M is an $(n-2)\times(n-2)$ real orthogonal matrix. The authors prove that: (1) $B-L$ can be decomposed as the topological product $C\times(G/G')$, where C is the open unit disc; (2) for a

given real-valued continuous function f on L, the Poisson integral $u(z)=\int f(x)P(z,x)\dot{x}$, where $\dot{x}$ is the volume element of L, is continuous on $R\cup B$, harmonic on R, and harmonic in its C-coördinate. *F. D. Quigley* (New Orleans, La.)

148:

Docquier, Ferdinand. Holomorphe Ausdehnung komplexer Mannigfaltigkeiten und Approximation holomorpher Funktionen. Schr. Math. Inst. Univ. Münster **13** (1958), 49 pp.

In der klassischen Funktionentheorie gibt der Rungesche Satz die Möglichkeit, in gewissen Gebieten G der z-Ebene $C(z)$ holomorphe Funktionen durch Funktionen zu approximieren, die noch in grösseren Gebieten $G^*\supset G$ holomorph sind. H. Behnke und K. Stein haben dieses Resultat auf die Funktionentheorie mehrerer Veränderlichen verallgemeinert. Dabei zeigte sich, dass es vernünftig ist, sich auf den Fall zu beschränken, wo G, G^* Holomorphiegebiete sind. Unter dieser Voraussetzung und der Annahme, dass G unverzweigt ist, erkannten sie als notwendige und hinreichende Bedingung für die Gültigkeit des Approximationssatzes die holomorphe (reguläre) Ausdehnbarkeit von G auf G^*. Der Verf. überträgt nun die grundlegenden Begriffe von H. Behnke und K. Stein auf den Fall, dass G, G^* allgemeine abstrakte Steinsche Mannigfaltigkeiten sind, und beweist für G, G^* den Behnke-Steinschen Satz. Dabei zieht er wesentlich die Okasche Einbettungsmethode und die Theorie komplexanalytischer Vektorraumbündel heran. Die für die Arbeit wichtigen Begriffe werden in einem einführenden Kapitel definiert. *H. Grauert* (Princeton, N.J.)

149:

Yano, Kentaro. On Walker differentiation in almost product or almost complex spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **61** = Indag. Math. **20** (1958), 573–580.

The paper is mainly concerned with establishing for an almost product space (a manifold with second order mixed tensor field F satisfying $F^2=I$) results analogous to those given by the reviewer for an almost complex space [C. R. Acad. Sci. Paris **245** (1957), 1213–1215; MR **19**, 680]. In particular, a number of differentiations are given which are analogous to torsional differentiation.

A. G. Walker (Liverpool)

SPECIAL FUNCTIONS

See also 36, 273.

150:

Arutyunyan, V. M.; Muradyan, R. M.; and Sokolov, A. A. An asymptotic expression for a degenerate hypergeometric function. Dokl. Akad. Nauk SSSR **122** (1958), 751–754. (Russian)

The asymptotic behavior of the Whittaker function $W_{k,m}(x)$ for $k\to\infty$ is given on $[0,\infty)$ when m is fixed. The authors appear to be unaware of the literature on this topic, for example, the book by H. Bucholz, *Die konfluente hypergeometrische Funktion* [Springer-Verlag, Berlin, 1953; MR **14**, 978]. *N. D. Kazarinoff* (Ann Arbor, Mich.)

151:

Obrechkoff, N. Sur quelques propriétés des zéros des polynomes classiques orthogonaux. C. R. Acad. Bulgare Sci. **10** (1957), 435–438. (Russian summary)

The author proves the following theorem concerning the Jacobi polynomials $P_n^{(a,b)}(x)$, where a and b are arbitrary real numbers greater than -1. Let the zeros x_k of $P_n^{(a,b)}(x)$ be labelled so that $x_1<x_2<\cdots<x_n$. If the number

$$N=(b^2-a^2)[(2n+a+b)(2n-2+a+b)]^{-1}$$

is situated between x_p and x_{p+1}, then $P_{n-2}^{(a,b)}(x)$ has one zero in each interval (x_k, x_{k+1}) for $k=1, 2, \cdots, p-1, p+1, \cdots, n-1$. If $N=x_q$, then $P_{n-2}^{(a,b)}(x)$ has one zero in each interval (x_k, x_{k+1}) for $k=1, 2, \cdots, q-2, q+1, \cdots, n-1$. The proof makes use of the partial fraction development for $[P_{n-2}^{(a,b)}(x)/P_n^{(a,b)}(x)]=\sum A_j(x-x_j)^{-1}$ in which the A_j are computed by use of various recursion formulas. A corresponding result is stated for the Laguerre and Hermite polynomials. *M. Marden* (Milwaukee, Wis.)

152:

Obrechkoff, Nikola. Sur les zéros des dérivées secondes des polynomes classiques orthogonaux. C. R. Acad. Bulgare Sci. **10** (1957), 439–442. (Russian summary)

Using the notation of the preceding article, the author shows that, if $P_n^{(a,b)}(c)\neq 0$, where $c=(b-a)/(a+b+2)$ and $x_p<c<x_{p+1}$, then the second derivative y'' of the Jacobi polynomial $y=P_n^{(a,b)}(x)$ has a zero between every two consecutive zeros of $P_n^{(a,b)}(x)$ except (x_p, x_{p+1}). If $P_n^{(a,b)}(c)=0$, the property holds except for the intervals (x_p, c) and (c, x_{p+1}). The proof uses the differential equation for y and the partial fraction development for y''/y. *M. Marden* (Milwaukee, Wis.)

153:

Engelis, G. K. Polynomials given by their Rodrigues formula. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 137–143. (Russian. Latvian summary)

Let $\alpha(x)$, $\beta(x)$ be polynomials of degrees s, $s+1$, and let $\rho(x)$ be a solution of $\rho'/\rho=\alpha/\beta$. The author considers the polynomials $P_n(x)=\rho^{-1}(d/dx)^n[\rho\beta^n]$. The author has previously [Uč. Zap. Latv. Gos. Univ. **8** (1956), 55–66] proved analogues of properties of Legendre polynomials $(s=1)$, including that $P_n(x)$ is orthogonal to $1, x, \cdots, x^{n-1}$ over (a, b) with weight-function $\rho(x)$ if $\rho(a)\beta(a)=\rho(b)\beta(b)=0$. This orthogonality-property, holding over several intervals, is now used to construct a determinantal representation of $P_n(x)$. Also found here are recurrence formulae, a generating function and integral representations, similar to those for the Legendre case.

F. V. Atkinson (Canberra)

154:

Mikolás, Miklós. On common characterization of the Jacobi, Laguerre and Hermite polynomials. Mat. Lapok **7** (1956), 238–248. (Hungarian. Russian and English summaries)

The polynomials indicated in the title are the only ones which form a (weighted) orthogonal system and satisfy at the same time a differential equation of the Sturm-Liouville type containing a parameter.

G. Szegő (Stanford, Calif.)

155:

Császár, Ákos. Sur les polynomes orthogonaux classiques. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 33–39.

Let (a, b) be finite or infinite, $w(x)>0$, $w'(x)/w(x)=L(x)/Q(x)$, $Q(x)>0$, $a<x<b$, where $L(x)$ is linear and $Q(x)$ of degree 2. Also let $\int_a^b w(x)dx$ be finite, $w(x)Q(x)x^k\to 0$ for $x\to a$ and $x\to b$ for all positive integers k. Under these conditions:

$$\text{(a)} \qquad p_n(x) = (w(x))^{-1}\left(\frac{d}{dx}\right)^n [w(x)(Q(x))^n]$$

represents a polynomial of degree n and $\{p_n(x)\}$ is an orthogonal system with weight $w(x)$ in (a, b); (b) $\{p_n^{(k)}(x)\}$ is an orthogonal system with weight $w(x)(Q(x))^k$; (c) $p_n(x)$ satisfies the differential equation

$$Q(x)y''+[L(x)+Q'(x)]y'+c_ny = 0, \quad c_n = \text{const.}$$

G. Szegö (Stanford, Calif.)

156:

Mezger, Fritz W. Evaluation of integrals arising in particle attenuation problems. J. Math. Phys. **37** (1958), 79–88.

Integrals of the type

$$J_{n,m}(\alpha, \delta) = \int_\delta^\infty y^m(y^2+\alpha^2)^{-\frac{1}{2}n}\exp[-(y^2+\alpha^2)^{\frac{1}{2}}]dy$$

with n and m restricted to be integral values ($\delta<\alpha<\infty$) are investigated. Recursion relations in the index n with the index m fixed and in the index m with the index n fixed and expressions for the derivatives of $J_{n,m}$ with respect to α are established. A reduction of $J_{n,m}$ to tabulated functions is given for the cases of arbitrary n with odd integer m and of arbitrary m with even integer n.

F. Oberhettinger (Madison, Wis.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 304, 440, 457, 641.

157:

Budak, B. M.; and Gorbunov, A. D. On the difference method of solution of the Cauchy problem for the equation $y'=f(x, y)$ and for the system of equations $x_i'=X_i\ (t, x_1, \cdots, x_n)$, $i=1, \cdots, n$ with discontinuous right-hand sides. Vestnik. Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 5, 7–12. (Russian)

The authors obtain local existence and uniqueness of the solution of (*) $y'=f(x, y)$, $y(x_0)=y_0$, and also the continuous dependence of solutions of $y'=f(x, y, \lambda)$ on initial conditions and the parameter in cases where $f(x, y)$ or $f(x, y, \lambda)$ is allowed to have certain discontinuities. The function $f(x, y)$ is defined and continuous in a rectangle R: $|x-x_0|\leq A$, $|y-y_0|\leq B$, except for a finite number of lines of discontinuity. These are a finite family of vertical line segments, $x=\text{const.}$, $|y-y_0|\leq B$, and a finite family of smooth nonintersecting curves $y=\varphi_i(x)$, $|x-x_0|\leq A$. Within each of the finite number of regions into which R is thus subdivided, $f(x, y)$ is continuous and satisfies a Lipschitz condition in y. Finally, along each curve $y=\varphi_i(x)$, the limiting values of $f(x, y)$ from the two sides satisfy

$$(^{**}) \quad [f(x, \varphi_i(x)+0)-\varphi_i'(x)][f(x, \varphi_i(x)-0)-\varphi_i'(x)] \geq \alpha > 0,$$

where α is a constant. With these hypotheses approximate solutions $\bar{y}(x, h)$ are defined on $|x-x_0|\leq\min(A, B/M)$ ($M=\max|f|$ on R) by the usual broken-line construction with step-length h. The approximations $\bar{y}(x, h)$ are shown to converge uniformly to a unique solution $y(x)$ on this interval. The method is very similar to standard techniques, a crucial point being to show that the approximating broken lines have only a finite number of points in common with the curves of discontinuity of $f(x, y)$. The system

$$x_i' = X_i(t, x_1, \cdots, x_n) \quad x_i(t_0) = x_i^0 \quad (i = 1, 2, \cdots, n)$$

is treated similarly. A condition similar to (**) is imposed on the limiting values of the X_i on the surfaces of discontinuity.

W. S. Loud (Minneapolis, Minn.)

158:

Yakubovič, V. A. Remarks on some papers on linear systems of differential equations with periodic coefficients. Prikl. Mat. Meh. **21** (1957), 707–713. (Russian)

Very simple proofs are given of the following theorems. I. Consider the system

$$(1) \qquad d^2y/dt^2+[C+P(t)]y = 0,$$

where $y=(y_1, \cdots, y_n)$, $C=\text{diag}(\omega_1^2, \cdots, \omega_n^2)$, $\omega_j>0$ distinct, $P(t)$ any real $n\times n$ matrix whose elements are periodic functions of t of period $T=2\pi/\omega$, L-integrable in $[0, T]$. Suppose (2) $\omega_i\pm\omega_j\neq m\omega$, $m=0, 1, \cdots$, and (3) either (α) $P(t)=P_{-1}(t)$, or (β) $P(-t)=P(t)$. Then there is a number $\varepsilon>0$ such that for any matrix P as above with (4) $\int_0^T\|P(t)\|dt<\varepsilon$, all solutions of (1) are bounded in $(-\infty, +\infty)$. II. The same for system

$$d^2y/dt^2+Q(t)dy/dt+[C+P(t)]y = 0,$$

where (γ) $Q(-t)=-Q(t)$, $P(-t)=P(t)$. III. The same as I for system (1), where $P(t)=P_0(t)+\Phi(t)$, $P_0=\text{diag}(P_1, \cdots, P_k)$, each matrix P_j of order n_j, $n_1+\cdots+n_k=n$, satisfying either (α) or (β), and all elements of Φ on or above the matrices P_j are null. [For another simple type of proof see J. K. Hale, Illinois J. Math. **2** (1958), 586–592; MR **20** #3323 (not quoted).] The author mentions that M. G. Kreĭn [Pamyati Aleksandra Aleksandroviča Andronova, pp. 413–498, Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 738] and I. M. Gelfand and V. B. Lidskiĭ [Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1 (63), 3–40 = Amer. Math. Soc. Transl. (2) **8** (1958), 143–181; MR **17**, 482; **19**, 960] have proved that condition (2) can be replaced by $\omega_i+\omega_j\neq m\omega$. Theorems I, II, III are improved forms of previous analogous theorems (λP, λQ replacing P, Q, and $|\lambda|<\varepsilon$ replacing (4)) proved by L. Cesari [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11** (1941), 633–695; MR **8**, 208], J. K. Hale [Riv. Mat. Univ. Parma **5** (1954), 137–167; MR **17**, 36], and R. A. Gambill [ibid., 169–181; **17**, 36], by a method of successive approximations which has been used extensively by Hale [Illinois J. Math. **1** (1957), 98–104; J. Math. Mech. **7** (1958), 163–171; **19**, 276, 1178] and Gambill [Riv. Mat. Univ. Parma **7** (1956), 311–319; MR **19**, 1178] for non-linear problems. By the same method R. A. Gambill has also proved criteria for unboundedness [ibid. **6** (1955), 37–43; MR **17**, 849] (not quoted).

L. Cesari (Baltimore, Md.)

159:

Watanabe, Tadashi. Introduction of the translational operator in frequency domain and treatment of certain linear differential equations. I, II. Mem. Coll. Sci. Univ. Kyoto. Ser. A **28** (1958), 293–326.

Consider a linear differential equation with periodically varying coefficients, (1) $\sum_{j=0}^{n} a_j(t)y^{(n-j)}(t)=x(t)$, in which the $a_j(t)$ are of the form

$$a_j(t) = \sum_{\alpha=0}^{\nu} \sum_{k=-N_{j,\alpha}}^{N_{j,\alpha}} A_k^{(j)}(\alpha)e^{ik\omega_\alpha t} \quad (j = 0, 1, 2, \cdots, n),$$

where the $A_k^{(j)}$ are constants. Let $Y(p)$ be the Laplace transform of $y(t)$, and define a translation operator $T(\omega_\alpha)$ by $T(\omega_\alpha)Y(p)=Y(p-i\omega_\alpha)$. On applying the Laplace transformation to both sides of (1) one obtains a relation of the form $S(T, p)Y(p)=X(p)$, where $S(T, p)$ is a polynomial in T and p. The author establishes various properties of operators of the form $S(T,p)$ and, in a formal way, obtains a power series expansion for the inverse of $S(T, p)$. Similar results are obtained for a system of ordinary linear differential equations with periodically varying coefficients, and several special cases are analyzed in detail. *L. A. Zadeh* (Berkeley, Calif.)

160:

Breus, K. A. On the reducibility of the standard system of differential equations with periodical coefficients. Dokl. Akad. Nauk SSSR **123** (1958), 21–24. (Russian)

The author considers Hamilton's canonical equations (1) $dq_r/dt=\partial\mathscr{H}/\partial p_r$, $dp_r/dt=-\partial\mathscr{H}/\partial q_r$ $(r=1, 2, \cdots, m)$, where the Hamiltonian $\mathscr{H}$ is a quadratic form in p_r and q_r with periodical coefficients:

$$\mathscr{H} = \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(\omega t)q_iq_j + \sum_{i,j=1}^{m} b_{ij}(\omega t)q_ip_j + \frac{1}{2}\sum_{i,j=1}^{m} c_{ij}(\omega t)p_ip_j$$

(ω is a large parameter). He solves the problem of finding a transformation of the variables p_r and q_r which transforms the equations (1) again into Hamilton's canonical equations, with the Hamiltonian being a quadratic form but with constant coefficients. *M. Zlámal* (Brno)

161:

Lillo, James C. Linear differential equations with almost periodic coefficients. Amer. J. Math. **81** (1959), 37–45.

The differential equations considered are of the form $x'=A(t)x+B(t)$, where $A(t)$ is an $n\times n$ matrix, $B(t)$ is an n vector, and their entries are real or complex valued almost periodic functions. Sufficient conditions are found for the existence of almost periodic solutions of this system. These conditions do not include the existence of almost periodic solutions of $x'=A(t)x$. For example, let the system be of the form $x'=(A+C(t))x+B(t)$ wherein $C(t)$ and $B(t)$ have real almost periodic entries and A is a real constant matrix. Let $\lambda_j=\alpha_j+i\eta_j$, $j=1, 2, \cdots, h$, denote the characteristic roots of A, $m=\min|\alpha_j|$, and p be the number of λ_j that are conjugate roots of A. Assume that the parameters have been changed so that $m\geqq 1$ and denote the multiplicity of λ_j by m_j. Let $c=\inf_{t,i,j}|c_{ij}(t)|$ and assume that for some $\varepsilon>0$, $m-\varepsilon>c\sum_{j=1}^{h}(p+m_j)(m_j)^2$. Then the system possesses a unique almost periodic solution whose module is contained in the module of $(C(t), B(t), \mathrm{Im}(\lambda_j))$. *W. R. Utz* (Columbia, Mo.)

162:

César de Freitas, A. Sur les équations différentielles linéaires à coefficients fonctions en escalier. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 161–176.

The author studies in great detail the solution of linear differential equations whose coefficients are Heaviside distribution functions ("jump functions"). *R. Bellman* (Santa Monica, Calif.)

163:

César de Freitas, A. La théorie des distributions et le calcul symbolique des électrotechniciens dans le cas des circuits à constante concentrées. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 193–264. (Portuguese. French summary)

The author continues the investigations begun in the paper reviewed above, using the Laplace transform. *R. Bellman* (Santa Monica, Calif.)

164:

Krzywicka, E. On the solutions of the differential equation $x^{(n)}+A(t)x=0$ satisfying conditions at several points. Prace Mat. **2** (1958), 337–351. (Polish. Russian and English summaries)

Let $a_1, \cdots, a_r$ be real numbers with $a_1<a_2<\cdots<a_r$. At a_i the values of q_i derivatives of order less than r of $x(t)$ are given (the zero order derivative may be included). $\sum_{i=1}^{r} q_i=n$. If for the choice of q_i for every i and every collection $a_1, \cdots, a_r$ the equation $x^{(n)}+A(t)x=0$ (for $A(t)$ continuous) has a unique solution x such that the corresponding q_i derivatives of x take the prescribed values, the author says that the condition of the type $(q_1, \cdots, q_r)$ determines the solution uniquely.

The author shows that if $(q_1, \cdots, q_r)$ determines the unique solution for a certain choice of orders and values of q_i derivatives at a_i, then for a different choice of orders and values of the same number (q_i) of derivatives it also does.

Further he proves that, for $A(t)>0$, $(q_1, \cdots, q_r)$ determines the solution uniquely if $q_2, \cdots, q_r$ are even and, for $A(t)<0$, if $q_2, \cdots, q_{r-1}$ are even and q_r is odd.

He also shows that for $n\leqq 6$ the above conditions are necessary. The proofs are based on theorems from an unpublished paper by J. Mikusiński. *C. Masaitis* (Havre de Grace, Md.)

165:

Levinson, N. A boundary value problem for a singularly perturbed differential equation. Duke Math. J. **25** (1958), 331–342.

This paper extends a similar problem for second order equations [S. Haber and N. Levinson, Proc. Amer. Math. Soc. **6** (1955), 866–872; MR **17**, 618] to third and higher order equations. It is shown that a unique solution of the boundary value problem $\varepsilon y'''=f(x, y, y', y'', \varepsilon)$, $y(0)=a$, $y(1)=b$, $y'(1)=b_1$, say $y=\varphi(x, \varepsilon)$, exists for ε sufficiently small provided the reduced equation $(\varepsilon=0)$ satisfies conditions (A), and tends to a limit as $\varepsilon\to 0^+$ with the uniformity properties (B).

Conditions A: $f(x, u, u', u'', 0)=0$ possesses a solution $u=g(x)$ for $0\leqq x\leqq x_0$, $u=h(x)$ for $x_0\leqq x\leqq 1$, where $0<x_0<1$, $g(0)=a$; $h(1)=b$, $h'(1)=b_1$; further, $u(x)$ is of class $C^1(x)$, while $u''(x)$ is discontinuous at x_0. For $f(x, y, w, z, \varepsilon)$ it is also required, in the case under consideration, that

$$f_z(x, g, g', g'', 0) > 0 \qquad (0 \leqq x \leqq x_0),$$
$$f_z(x, h, h', h'', 0) < 0 \qquad (x_0 \leqq x \leqq 1),$$
$$f(x_0, g(x_0), 0, z, 0) > 0 \qquad (-\mu < z < \mu),$$

where the normalization

$$-g''(x_0) = h''(x_0) = \mu > 0, \quad g'(x_0) = h'(x_0) = 0$$

has been made. Finally, it is assumed that the solution of

$$f_z(x, g, g', g'', 0)\frac{d^2r}{dx^2} + f_w(x, g, g', g'', 0)\frac{dr}{dx} + f_y(x, g, g', g'', 0)r = 0,$$

$$r(x_0) = 0, \quad r'(x_0) = 1$$

for the interval $0 \leq x \leq x_0$, has a non-zero value at $x=0$.

Properties B: With $U(x)=g(x)$ for $0 \leq x \leq x_0$ and $U(x)=h(x)$ for $x_0 \leq x \leq 1$, $\varphi(x, \varepsilon)$, $\varphi'(x, \varepsilon)$ tend uniformly to $U(x)$, $U'(x)$ over $[0, 1]$, and $\varphi''(x, \varepsilon) \to U''(x)$ uniformly over $[0, x_0-\delta_0]$ and $[x_0+\delta_0, 1]$, for any $\delta_0 > 0$, as $\varepsilon \to 0^+$.

G. E. Latta (Stanford, Calif.)

166:

★**Slotnick, D. L. Asymptotic behavior of solutions of canonical systems near a closed, unstable orbit.** Contributions to the theory of non-linear oscillations, Vol. IV, pp. 85–110. Annals of Mathematics Studies, no 41. Princeton University Press, Princeton, N.J., 1958. ix+211 pp. \$3.75.

Non-linear Hamiltonian equations are considered which involve t explicitly, with period 2π, and for which on the linear approximation the solution gives oscillations about O with period $2q\pi/p$ (p, q integers). Under suitable general hypotheses (which usually are not fully explicit) it is shown that O is actually unstable, and that orbits asymptotic from it can be specified via an invariant curve which is analytic except at O and hyper-continuous at O. In this the essential preliminary step is to transform the equations to the form

$$\dot{u} = bu^{s-1} + \cdots, \quad \dot{v} = -b(s-1)u^{s-2}v + \cdots$$

by use of coordinates u, v which on the linear approximation are constant; $v=O(u^2)$ on the invariant curve.

T. M. Cherry (Melbourne)

167:

Švec, Marko. Sur le comportement asymptotique des intégrales de l'équation différentielle $y^{(4)}+Q(x)y=0$. Czechoslovak Math. J. **8** (**83**) (1958), 230–245. (Russian summary)

It is assumed that $Q(x)$ is continuous and non-negative, that it does not vanish on any interval, and that the solutions are oscillatory. Properties are deduced for a variety of expressions involving certain solutions and their derivatives. It is then shown that the equation has two solutions which approach zero at infinity and, for certain bounds on $Q(x)$, that the equation also has unbounded solutions. Other results pertain to the existence or otherwise of integrals such as $\int^\infty y'^2dx$, for example.

T. E. Hull (Pasadena, Calif.)

168:

Makaeva, G. S. The asymptotic behavior of solutions to differential equations involving a small parameter, whose systems of "rapid motions" are nearly Hamiltonian. Dokl. Akad. Nauk SSSR **121** (1958), 973–976. (Russian)

In 1957 L. S. Pontryagin posed the following problem: Study the system

$$(*) \quad \varepsilon\frac{dx}{dt} = \frac{\partial H(\mathbf{p})}{\partial y} + \varepsilon X(\mathbf{p}, \varepsilon), \quad \varepsilon\frac{dy}{dt} = -\frac{\partial H(\mathbf{p})}{\partial x} + \varepsilon Y(\mathbf{p}, \varepsilon),$$

$$\frac{dz_j}{dt} = Z(\mathbf{p}, \varepsilon) \quad (j = 1, \cdots, l; \mathbf{p} = (x, y, z_1, \cdots, z_l)),$$

and derive from it the results of V. M. Volosov [Mat. Sb. (N.S.) **31** (**73**) (1952), 645–686; MR **14**, 1086]. The present note is devoted to this end. It is assumed that the Hamiltonian system obtained by replacing t in (*) by $\varepsilon\tau$ and then setting $\varepsilon=0$ has a closed phase trajectory. Neighboring closed phase trajectories of (*) are studied under assumptions of regularity on H, X, Y and Z; and the functional dependence of x and y is determined.

N. D. Kazarinoff (Ann Arbor, Mich.)

169:

Kuzmak, G. E. Asymptotic solutions of the equation of motion for a non-linear oscillatory system having one degree of freedom and slowly varying parameters. Dokl. Akad. Nauk SSSR **120** (1958), 461–464. (Russian)

The author considers the equation

$$(*) \quad \frac{d^2y}{dt^2} + \varepsilon f(\varepsilon t, y)\frac{dy}{dt} + F(\varepsilon t, y) = 0,$$

with ε a small parameter and $f(\tau, y)$, $F(\tau, y)$ sufficiently smooth for $0 \leqq \tau \leqq \tau_0$, $|y| \leqq y$. Assuming that there is a solution of oscillatory type, he shows how an expression $\bar{y}(t)$ may be constructed which satisfies (*) on $[0, \tau_0/\varepsilon]$, for small enough $\varepsilon > 0$, to within terms of order ε^2. Use is made of results of Dorodnicin [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6 (52), 3–96; MR **14**, 876] and the author [Prikl. Mat. Mech. **21** (1957), 262–271; MR **19**, 651].

H. A. Antosiewicz (Los Angeles, Calif.)

170:

Volosov, V. M. Asymptotic theory of the integrals of some disturbed systems. Dokl. Akad. Nauk SSSR **121** (1958), 959–962. (Russian)

Consider the undisturbed system (*): $\dot{x}_0 = M(x_0, y_0, \mu_0)$, $\dot{y}_0 = N(x_0, y_0, \mu_0)$, $(\mu_0 = \{\mu_{10}, \cdots, \mu_{n0}\})$, with M and N sufficiently regular to guarantee existence and uniqueness of solutions and their continuous dependence upon μ_0 and initial conditions. It is further assumed that: (1) $\Phi(x_0, y_0, \mu_0)$ is a sufficiently smooth time independent integral of (*); (2) there exists a family of periodic solutions of (*) initiating in some region $\bar{G}$, satisfying the equation $\Phi(x_0, y_0, \mu_0) = c_0$, which may be written as $x_0 = x_0(c_0, \mu_0, \omega(c_0, \mu_0)t+h)$, $y_0 = y_0(c_0, \mu_0, \omega(c_0, \mu_0)t+h)$, where h is an arbitrary constant, $T(c_0, \mu_0)$ is the period belonging to the orbit $\Phi(x_0, y_0) = c_0$, and ω is the frequency; (3) x_0 and y_0 are periodic in $\omega t+h$ with period 2π and $\omega \geqq \alpha > 0$. The investigation concerns integrals of the disturbed system

$$\dot{x} = M(x, y, \mu) + \varepsilon f^{(y)}(x, y, \mu),$$

$$\dot{y} = N(x, y, \mu) - \varepsilon f^{(x)}(x, y, \mu), \quad \dot{\mu} = \varepsilon\varphi(x, y, \mu),$$

where μ and φ are n-vectors and ε is a small parameter. The functions $f^{(x)}$, $f^{(y)}$ and φ are taken to be sufficiently regular.

If x_0, y_0, and μ_0 are replaced in $\Phi(x_0, y_0, \mu_0) = c_0$ by x, y, and μ, which satisfy the disturbed system, then Φ becomes a function $c(t, \varepsilon)$. The problem studied is the asymptotic behavior of c on certain intervals of t which

become infinite as $\varepsilon \to 0^+$. The main results are integral formulas for $\dot{c}$ and $\dot{\mu}$:

$$\dot{c} = \varepsilon T^{-1} \oint \lambda(f^{(x)}dx_0 + f^{(y)}dy_0) + \varepsilon T^{-1} \int_T \varphi(\nabla_\mu \Phi)dt;$$

$$\dot{\mu} = \varepsilon T^{-1} \int_T \varphi dt;$$

where λ is an integrating factor for (*).

N. D. Kazarinoff (Ann Arbor, Mich.)

171:

Glazman, I. M. Oscillation theorems for differential equations of higher orders and the spectrum of the respective differential operators. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 423–426. (Russian)

This paper contains the statement of a number of results extending oscillation theorems of Weyl, Rapoport, Hille, Bellman, and the author to equations of degree >2. The equations considered are

$$(1) \qquad \sum_{k=0}^{n} (-1)^{n-k}[p_k(x)y^{(n-k)}]^{(n-k)} = \lambda y \qquad (p_0(x) = 1,\ 0 \leqq x < \infty);$$

$$(2) \qquad (-1)^n y^{(2n)} + q(x)y = \lambda y \quad (0 \leqq x < \infty).$$

Oscillation is defined to mean that the solutions have arbitrarily large zeroes of multiplicity n. For $\lambda = \lambda_0$, oscillation of (1) is equivalent to the part of the spectrum of T to the left of λ_0 being finite, where T is any self-adjoint operator generated by (1). Five theorems consider various cases for $q(x)$ in (2). Methods of proof are sketched.

S. Hoffman (Hartford, Conn.)

172:

Minorsky, Nicolas. Sur la synchronisation. C. R. Acad. Sci. Paris **247** (1958), 705–708.

Conditions for the non-linear differential equation of Van der Pol,

$$\ddot{x} - (a - cx^2)\dot{x} + x = e \sin \omega t,$$

to have periodic solutions, with the same period ω as the forcing term, are found by a perturbation method.

P. Franklin (Cambridge, Mass.)

173:

Miščenko, E. F. Asymptotic theory of relaxation oscillations described by systems of second order. Mat. Sb. N.S. **44 (86)** (1958), 457–480. (Russian)

The system considered is $\{\varepsilon\dot{x} = f(x, y);\ \dot{y} = g(x, y)\}$, where ε is a small positive parameter, f and g are non-linear, sufficiently regular functions, and x and y are real. Away from the curve in the phase plane where $f(x, y) = 0$, it is easy to solve the reduced system with $\varepsilon = 0$. In this case, x is rapidly varying in comparison with y. Near the curve where $f(x, y) = 0$, one must first solve the original system and then let $\varepsilon \to 0^+$ to obtain the behavior of the solution of the reduced system. Here one finds that x is slowly varying. The equation $f(x, y_0)$ isolates, in the phase plane, the totality of positions of equilibrium for $\varepsilon\dot{x} = f(x, y_0)$. The curve $f(x, y) = 0$ represents the set of all such positions. Along the portion of a trajectory on which x is slowly varying the equilibrium is stable; at the curve where $f = 0$ there is a discontinuity and the equilibrium is unstable. The present paper constitutes a development of the author's joint work with Pontryagin [Dokl. Akad. Nauk SSSR **102** (1955), 889–891; MR **17**, 153].

The behavior of a solution of the original system when x is slowly varying, and also near a point of discontinuity, is found up to terms of order ε^{n+1} $(n = 0, 1, \cdots)$. If the reduced system has a stable discontinuous limiting solution Z_0 with period T_0, then near it the original system has a periodic cycle of period

$$T = T_0 + \varepsilon^{2/3}Q_1 + \varepsilon \ln \varepsilon Q_2 + \varepsilon Q_3 < l(\varepsilon^{7/6}).$$

An algorithm is given for the computation of the constants Q_i.

N. D. Kazarinoff (Ann Arbor, Mich.)

174:

★DeVogelaere, René. On the structure of symmetric periodic solutions of conservative systems, with applications. Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 53–84. Annals of Mathematics Studies, no 41. Princeton University Press, Princeton, N. J., 1958. ix+211 pp. $3.75.

For a system $\ddot{x} = U_x(x, y)$, $\ddot{y} = U_y(x, y)$ in which U is an even function of y, periodic solutions are considered for which the energy $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - U$ has a given value and which (1) intersect Ox perpendicularly or (2) contain a point where $\dot{x} = \dot{y} = 0$. By numerical integration one-parameter families of solutions satisfying (1) or (2) can be computed, and the detection amongst these of periodic solutions is helped by obvious considerations of symmetry and reversibility. The paper begins by formalizing these considerations in the context of surface transformations, and concludes with numerical results for Störmer's well-known electron orbit equations, marshalled and classified in a natural manner.

T. M. Cherry (Melbourne)

175:

Frommer, M. Die Integralkurven einer gewöhnlichen Differential-gleichung erster Ordnung in der Umgebung rationaler Unbestimmtheitsstellen. Advancement in Math. **3** (1957), 85–126. (Chinese)

This is a Chinese translation of Frommer's article of the above title which appeared originally in Math. Ann. **99** (1928), 222–272.

Choy-tak Taam (Washington, D.C.)

176a:

Mitropol'skiĭ, Yu. A. On the investigation of an integral manifold for a system of nonlinear equations with variable coefficients. Ukrain. Mat. Ž. **10** (1958), 270–279. (Russian. English summary)

176b:

Mitropol'skiĭ, Yu. A. On the stability of a one-parameter family of solutions of a system of equations with variable coefficients. Ukrain. Mat. Ž. **10** (1958), 389–393. (Russian. English summary)

In a recent book, hereafter referred to as (*), by N. Bogoliubov and Yu. A. Mitropol'skiĭ [*Asimptotičeskie metody v teorii nelineĭnyh kolebaniĭ*, Gosudarstv. Izdat. Tehn. Teor. Lit., Moscow, 1955; MR **17**, 368; revised 1958] a detailed discussion is given of the solutions of the equation

$$(1) \qquad dg/dt = \omega + P(t, g, h, \varepsilon),\ dh/dt = Hh + Q(t, g, h, \varepsilon)$$

where $\omega > 0$, ε are real parameters, g is a scalar, h is an $(n-1)$-vector, the eigenvalues of the constant matrix H

have nonzero real parts, P, Q are Lipschitzian in g, h, continuous in t, periodic in g of period π, and $P=O(\|h\|+|\varepsilon|)$, $Q=O[(\|h\|+|\varepsilon|)^2]$ as $\|h\|\to 0$, $\varepsilon\to 0$. In particular, it is shown that for ε sufficiently small there exists a unique two-dimensional integral manifold S of (1) defined parametrically by $h=f(t, g, \varepsilon)$, where $f(t, g, \varepsilon)=f(t, g+\pi, \varepsilon)$. The stability of S is also discussed. If, in addition, P, Q are almost periodic in t uniformly with respect to g, h, then $f(t, g, \varepsilon)$ is almost periodic in t uniformly with respect to g. Systems of equations of type (1) arise in many applications; e.g., in the problem of the perturbation of an autonomous differential system which has a stable periodic solution [N. Levinson, Ann. of Math. (2) **52** (1950), 727–728; MR **12**, 335; Y. A. Mitropolski, Ukrain. Mat. Ž. **9** (1957), 296–309; MR **19**, 960, S. P. Diliberto and G. Hufford, *Contributions to the theory of nonlinear oscillations*, vol. 3, pp. 207–236, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 653] and in the method of averaging of N. Krylov and N. Bogoliubov [see, e.g. (*), or the translation by S. Lefschetz of Krylov and N. Bogoliubov's *Introduction to non-linear mechanics*, Ann. Math. Studies, no. 11, Princeton Univ. Press, 1943; MR **4**, 142]. The remarkable thing about the book (*) is that the method of proof allows one to show the existence of the manifold S without assuming any special properties about the dependence of P and Q upon t.

In the first paper being reviewed here, the author uses the same method of proof as in (*) to prove the existence and asymptotic stability of an integral manifold S of (1) as above for the case where (2) $\omega=\omega(t)$, $H=H(t)$ are continuous functions of t and the matrix solution $U(t, s)$ of the equation $dh/dt=H(t)h$, $U(s, s)=E$, the identity matrix, satisfies the property $\|U(t, s)\|\leqq Ke^{-\gamma(t-s)}$, $t\geqq s$, $K>0$, $\gamma>0$.

The second paper is an application of the first paper to the equation

$$(3)\qquad dx/dt = X(\tau, x)+\varepsilon X^*(\tau, \theta, x, \varepsilon), \quad \tau = \varepsilon t,$$
$$d\theta/dt = \nu(\tau) > 0,$$

X^* periodic in θ of period π, and the equation $dx/dt=X(\tau, x)$, where τ is considered a parameter, $-\infty<\tau<+\infty$, and has an asymptotically orbitally stable periodic solution for every fixed τ. By introducing new coordinates in a neighborhood of x_0, it is shown that the new variables satisfy an equation of type (1) satisfying (2). By applying the results of the first paper, the existence of a stable integral manifold of (3) is proved.

For some other applications of the methods of (*) see O. B. Lykova [Ukrain. Mat. Ž. **9** (1957), 281–295, 419–431; MR **19**, 959, 857]. *J. K. Hale* (Baltimore, Md.)

177:

Lillo, James C. On almost periodic solutions of differential equations. Ann. of Math. (2) **69** (1959), 467–485.

Consider the real non-linear differential system

$$z' = F(z, \mu, t), \quad z' = dz/dt,$$

and assume that for $\mu=\mu_0$ the system possesses a stable almost periodic solution $p(\mu_0, t)$. If $F(z, \mu, t)$, for each fixed μ, is almost periodic in t uniformly with respect to z in a cylindrical neighborhood of $p(\mu_0, t)$ of positive radius, then for μ near μ_0 there exists a unique stable almost periodic solution $p(t, \mu)$ near $p(t, \mu_0)$. This perturbation problem is solved by a geometric approach based on properties of translation numbers. The approach is then applied to the generalized pendulum equation

$$x'' + \mu f(x)x' + g(x) = 0,$$

where $x'=dx/dt$, $f(x)$ and $g(x)$ are real almost periodic functions and μ is a real parameter. The results are the almost periodic generalizations of results secured by G. Seifert [*Contributions to the theory of nonlinear oscillations*, vol. 3, pp. 1–16, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 305] in his study of this equation with $g(x)$ and $f(x)$ periodic. Finally, the Riccati equation $x'+q(t)x^2+p(t, \mu)=0$, where $q(t)$ and $p(t, \mu)$ are real almost periodic functions for each value of the real parameter μ, is shown to have almost periodic solutions, and these results are compared with those of L. Markus and R. Moore [Acta Math. **96** (1956), 99–123; MR **18**, 306]. *W. R. Utz* (Columbia, Mo.)

178:

Škil', M. I. On the asymptotic representation of solutions of a system of ordinary linear differential equations. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 123–127. (Ukrainian. Russian and English summaries)

Solutions of the system $\dot{x}=A(\tau, \varepsilon)x+\varepsilon B(\tau, \varepsilon)e^{i\theta(t)}$ ($\dot{x}=dx/dt$), where A is a real 4×4 matrix, B is a 4-vector, $\tau=\varepsilon t$, $(0\leqq\tau\leqq L)$, $A(\tau, \varepsilon)=\sum_0^\infty \varepsilon^s A^s(\tau)$, $B(\tau, \varepsilon)=\sum_0^\infty \varepsilon^s B^s(\tau)$, and ε is a small parameter, are described. Hypotheses are that the characteristic roots of $A^0(\tau)$ are $\lambda_1=\lambda_2=i\alpha(\tau)$, $\alpha(\tau)>0$ and $\lambda_3=\lambda_4=-i\alpha(\tau)$, and that either $i\dot{\theta}=\lambda_1$ for some τ on $[0, L]$ or this happens for no τ on $[0, L]$.
N. D. Kazarinoff (Ann Arbor, Mich.)

179:

Šiškin, A. A. Eigenvalues and characteristic functions of some fourth order differential operators containing a small parameter in the higher derivative term. Dokl. Akad. Nauk SSSR **123** (1958), 249–251. (Russian)

Let

$$L_\eta = -\eta(d^2/dx^2)(p^2(x)d^2/dx^2)+L_0,$$
$$L_0 = -(d/dx)(p_1(x)d/dx)$$

be two differential operators considered on [0, 1], where $p_i(x)$ $(i=1, 2)$ are positive functions i-times continuously differentiable. The author studies the boundary-value problems

$$(\mathrm{I}_\sigma)\quad L_\eta(u) = \lambda u,\ u(0) = u(1) = 0, \quad u^{(\sigma)}(0) = u^{(\sigma)}(1) = 0$$
$$(\sigma=1, 2;\ \text{small values of } \eta);$$

$$(\mathrm{II})\qquad L_0(y) = \lambda y, \quad y(0) = y(1) = 0.$$

The problem $L_\eta(u)=0$, $u(0)=u(1)=0$, $u^{(\sigma)}(0)=u^{(\sigma)}(1)=0$ has non-trivial solutions for a discrete set $\{\eta_{\sigma,n}\}$, where $\eta_{\sigma,n}$ are of the form $\nu/((3-\sigma)n\pi)$, $n=1, 2, \cdots$, $\sigma=1, 2$. Introducing a stability domain R_σ for (I_σ), formed by a union of intervals not containing a certain neighbourhood of $\eta_{\sigma,n}$ $(n=1, 2, \cdots)$, the author gives for $\eta\in R$ precise evaluations of the solutions of nonhomogeneous equation (I_σ), of Green's function and finally of the eigenvalues of (I_σ), i.e.

$$\lambda_k(\eta) = \lambda_k - \eta^\sigma\phi_{\sigma,k}+\eta^{\sigma+1}O(1),$$

where $\lambda_k(\eta)$ is the kth positive eigenvalue of (I_σ), λ_k the corresponding one of (II) and $\sigma=1, 2$. No proof is given.
C. Foiaş (Bucharest)

180:

Denjoy, Arnaud. Sur les courbes définies par les équations différentielles. Advancement in Math. **4** (1958), 161–187. (Chinese)

This is a Chinese translation of A. Denjoy's article "Sur les courbes définies par les équations différentielles à la surface du tore" which appeared originally in J. Math. Pures Appl. (9) **11** (1932), 333–375.

Choy-tak Taam (Washington, D.C.)

181:

Rehlickiĭ, Z. I. Boundedness tests for solutions of linear differential equations with variable lag of argument. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 447–449. (Russian)

The author states without proof a number of results concerning the boundedness of solutions of equations of the form $y'(t)=A(t)y(t-a(t))=x(t)$ in the case where $A(t)$ approaches a constant as t increases.

R. Bellman (Santa Monica, Calif.)

182:

Kamenskiĭ, G. A. On the general theory of equations involving a deviating argument. Dokl. Akad. Nauk SSSR **120** (1958), 697–700. (Russian)

The author states without proof four theorems concerning the existence, uniqueness, smoothness, and continuous dependence on initial data of solutions of systems of ordinary differential equations with deviating arguments. *H. A. Antosiewicz* (Los Angeles, Calif.)

183:

Korobeĭnik, Yu. F. On equations of infinite order with polynomial coefficients. Dokl. Akad. Nauk SSSR **122** (1958), 339–342. (Russian)

The (differential) equations in question have the form (*) $\sum_{i=0}^{\infty} P_i(x)y^{(i)}(x)=f(x)$, with $\{P_i(x)\}$ polynomials. (*) is called regular if: (1) $P_0(x)\equiv a_0\neq 0$; (2) $P_i(x)$ is of degree $\leqq i-1$ $(i>0)$; (3) $f(x)$ is an entire function. Six theorems are given, and it is stated that the method of proof consists in transforming (*) to an infinite system of linear algebraic equations. A sample result: Let (*) be regular, with $a_0=1$, $P_k(x)=\sum_{s=0}^{k-1} a_k{}^s x^s$ $(k>0)$; let $\{A_k{}^s\}$ majorize $\{a_k{}^s\}$. Choose the positive sequence $\{A_k\}$ so that

$$\limsup_{k\to\infty} A_k^{-1}\sum_{m=1}^{k-1} m!A_m \sum_{s=0}^{m} A^s_{k-m+s}/(m-s)! < 1;$$

$$\limsup_{k\to\infty} A_k^{-1}\sum_{m=0}^{k-1} A_m < \infty; \quad \limsup_{k\to\infty} A_k{}^0/A_k < \infty.$$

Let K_A denote the class $\{f(x)\}$ of entire functions for which $\sum_{k=0}^{\infty} |f^{(k)}(0)A_k|<\infty$. For each $f(x)\in K_A$, equation (*) has a unique solution $y(x)$ in K_A, and for this solution the left side of (*) converges uniformly in every bounded region. Moreover, a constant D exists, independent of $f(x)$ and $y(x)$, such that $\sum_{k=0}^{\infty} |y^{(k)}(0)|A_k\leqq D\sum_{k=0}^{\infty} |f^{(k)}(0)|A_k$.

The non-regular case of (*) is briefly discussed, especially for the case where each $P_i(x)$ is of degree not exceeding one. *I. M. Sheffer* (University Park, Pa.)

184:

Melamed, E. Ya. Kennzeichen der Beschränkung der Lösungen von einigen partiellen differentialen Randproblemen im Banachraum. Ukrain. Mat. Ž. **10** (1958), 394–404. (Russian. German summary)

In dieser Arbeit wird ein System der Art

$$\begin{cases} \dfrac{\partial U}{\partial t} = A\,\dfrac{\partial U}{\partial x}+f(x,t) \\ U(x,0) = \varphi(x) \end{cases}$$

erörtert, wo $U(x,t)$ eine Vektor-Funktion mit den Werten im Banachraum $\mathscr{E}$ ist von der komplexen Variable x $(\operatorname{Im} x\geqq 0)$ und t $(0\leq t\leq T)$ beschränkt; A ist ein linear beschränkter Operator im Raum $\mathscr{E}$.

Es werden die Spektralbedingungen aufgestellt, bei denen den beschränkten Vektor-Funktionen $f(x,t)$ und $\varphi(x)$ eine beschränkte Lösung $U(x,t)$ des Systems entspricht. *Zusammenfassung des Autors*

PARTIAL DIFFERENTIAL EQUATIONS

See also 103, 134, 282, 437.

185:

Roždestvenskiĭ, B. L. On the Cauchy problem for quasilinear equations. Dokl. Akad. Nauk SSSR **122** (1958), 551–554. (Russian)

This paper gives a method of reducing the question of uniqueness of the generalized solution of the Cauchy problem for a system of quasilinear equations of the form

$$\frac{\partial u_i}{\partial t}+\frac{\partial \phi_i}{\partial x}(u_1,\cdots,u_n,t,x) = 0 \quad (i=1,2,\cdots,n)$$

to the question of uniqueness of continuous solutions of the Cauchy problem for a certain system of non-linear equations. This reduction enables the author to give a proof of a theorem on the uniqueness of the solution for a system of quasilinear equations. The method is illustrated by an example showing the uniqueness of the generalized solution of the problem of Cauchy for a single quasilinear equation. *C. G. Maple* (Ames, Iowa)

186:

Roždestvenskiĭ, B. L. Uniqueness of the generalized solution of the Cauchy problem for systems of quasilinear hyperbolic equations. Dokl. Akad. Nauk SSSR **122** (1958), 762–765. (Russian)

A system of functions $u_i(t,x)$ $(i=1,\cdots,n)$ which assume given values on the initial line $t=0$ and which satisfy the integral conditions

$$(*) \qquad \oint_C u_i(t,x)dx-\phi_i(u(t,x),t,x) = 0,$$

where C is an arbitrary piecewise smooth closed contour in the half-plane $t\geqq 0$, constitutes a generalized solution of the Cauchy problem for the conservative system of quasilinear equations

$$(**) \qquad \partial u_i/\partial t+\partial\phi_i(u,t,x)/\partial x = 0, \quad u=\{u_1,\cdots,u_n\}.$$

The existence of a solution to (*), however, is not a sufficient condition for the uniqueness of the generalized solution, which may be discontinuous. In an earlier paper [reviewed above] the author has studied the uniqueness of the generalized solution of the Cauchy problem for (**) for the case $n=1$ by considering an equivalent problem in the "potential" Φ_1 of u_1, namely that of finding continuous solutions of the Cauchy problem for a non-linear

equation satisfied by Φ_1. In the present paper he uses the same method to establish sufficient conditions for the uniqueness for any n. These are essentially that the discontinuities in the generalized solution must satisfy the shock criterion given by P. D. Lax [Comm. Pure Appl. Math. **10** (1957), 537–566; MR **20** #176], i.e., that the discontinuities be bounded above and below by successive eigenvalues of the system (**).

R. N. Goss (San Diego, Calif.)

187:

Burčuladze, T. V. Fundamental solutions of a system of differential equations. Soobšč. Akad. Nauk Gruzin. SSR **20** (1958), 391–398. (Russian)

A method of construction of an explicit solution of the differential equation

$$\sum_{l=1}^{2}\sum_{i=1}^{2}\sum_{k=1}^{2} A_{ik}^{jl}\frac{\partial^2 u_l}{\partial x_i \partial x_k}+\omega^2 u_j = 0,$$

where A_{ik}^{jl} and ω are constants and $\mathbf{u}=(u_1, u_2)$ is an unknown vector, is presented.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

188:

Berezanskiĭ, Yu. M. The uniqueness theorem in the inverse problem of spectral analysis for the Schrödinger equation. Trudy Moskov. Mat. Obšč. **7** (1958), 1–62. (Russian)

Let $L[u]=-\Delta u+c(p)u$ be the Schrödinger operator (where $c(p)$ is a real function twice continuously differentiable and p is a point of the space (R^2 or R^3 as $L[u]$ is considered in R^2 or R^3)). Let $\mathfrak{H}$ be the Hilbert space of the square integrable functions on G (G being a domain with regular boundary Γ, or the whole space, i.e., $\Gamma=0$) and $\mathfrak{D}$ be the space of those functions u of $\mathfrak{H}$ which are twice continuously differentiable with compact support, so that $\partial u/\partial n+\sigma(p)u=0$ on Γ (if $\Gamma\neq 0$), where $\sigma(p)$ is a real continuous function. L defined on $\mathfrak{D}$ can be prolonged to a selfadjoint operator A. Let E_λ be the spectral measure of A. Then [as was shown by T. Carleman Ark. Mat. Astr. Fys. (11) **24** (1934), 1–7] $(E_\lambda f)(p)=\int_G \vartheta(p, q, \lambda)f(q)dq$, and the kernel is absolutely continuous with regard to a non-decreasing function $\rho(\lambda)$, i.e. $d_\lambda\vartheta(p, q, \lambda)=\psi(p, q, \lambda)d\rho(\lambda)$, where the function $\psi(p, q, \lambda)$ is continuous in (p, q), twice differentiable in p (or in q), and verifies $L[u]=\lambda u$ and the boundary condition $\partial u/\partial n+\sigma(p)u=0$ (if $\Gamma\neq 0$). The inverse problem of the spectral analysis for $L[u]$ consists in finding $c(p)$ if $\vartheta(p, q, \lambda)$ with the above properties is given. In analogy with what happens in the case of ordinary differential equations [V. A. Marčenko, Trudy Moskov. Mat. Obšč. **1** (1952), 327–420; MR **15**, 315], M. G. Krein, I. M. Gelfand and B. I. Levitan raised (at the conference on differential equations in Moscow 1952) the question whether $c(p)$ is determined only by the boundary values of $\vartheta(p, q, \lambda)$. The affirmative answer to this question (i.e. the uniqueness of the inverse problem of the spectral analysis for Schrödinger's operator) is given by the author's basic theorem: If $G\neq R^n$ $(n=2, 3)$ and if Γ contains a plane region I on which $\sigma(p)=0$, the restriction to $I\times I\times(-\infty, +\infty)$ of $\vartheta(p, q, \lambda)$ uniquely determines $c(p)$ (in the class of sufficiently regular functions) and $\sigma(p)$ (on $\Gamma-I$). If $G=R^n$ $(n=2, 3)$ then $c(p)$ is uniquely determined by the matrix

$$\theta(p, q, \lambda) = \begin{Vmatrix} \vartheta(p, q, \lambda) & \dfrac{\partial\vartheta}{\partial n_p}(p, q, \lambda) \\[2ex] \dfrac{\partial\vartheta}{\partial n_q}(p, q, \lambda) & \dfrac{\partial^2\vartheta}{\partial n_p \partial n_q}(p, q, \lambda) \end{Vmatrix}$$

given for all $-\infty<\lambda<\infty$ and p, q on a regular surface contained in a region where $c(p)$ is already given. The difficult proof of this theorem is developed in § 2, and follows an ingenious method based on the remark that $2\lambda^{-1}\cdot\sin^2(\frac{1}{2}\sqrt{\lambda})t\cdot\psi(p, q, \lambda)$ satisfies in p (and in q also) the hyperbolic equation $\partial^2 u/\partial t^2+L[u]=\psi(p, q, \lambda)$ and $u(p, 0)=u_t'(p, 0)=0$. For this, the necessary facts about hyperbolic equations are established in § 1.

In the last paragraph the author studies (under the hypothesis that $G=R^3$, $c(p)=0$ for $|p|=(x^2+y^2+z^2)^{\frac{1}{2}}\geq R_0$) other inverse problems; for instance, let

$$u(p, q, k) = \frac{\exp(ik|p-q|)}{4\pi\,|p-q|}+v_c(p, q, k)$$

be the solution in p of $L[u]=k^2u$, $k>0$, where $|q|>R_0$ and

$$v_c(p, q, \lambda) = O(1/|p|),$$
$$\partial v_c(p, q, \lambda)/\partial|p|-ikv_c(p, q, \lambda) = o(1/|p|);$$

then if $v_c(p, q, \lambda)$ is given for all $k>0$, $|p|=|q|=R>R_0$, the coefficient $c(p)$ is uniquely determined. He shows that these problems are equivalent to the former inverse problem of the spectral analysis for $L[u]$.

C. Foiaş (Bucharest)

189:

Žislin, G. M. On the spectrum of Schrödinger's operator. Dokl. Akad. Nauk SSSR **122** (1958), 331–334. (Russian)

The author generalizes a theorem of an earlier note [same Dokl. **117** (1957), 931–934; MR **20** #1548], from an n-dimensional Laplace operator to a Schrödinger operator.

M. M. Day (Urbana, Ill.)

190:

Mihaĭlov, V. P. Analytic solution of the Goursat problem for a system of partial differential equations. Mat. Sb. N.S. **46** (88) (1958), 315–342. (Russian)

The author solves the problem of finding analytic solutions of the system

$$F_i(x, t, u_1, \cdots, u_n, p_1, \cdots, p_n, q_1, \cdots, q_n) = 0$$
$$(p_i = \partial u_i/\partial x,\ q_i = \partial u_i/\partial t,\ i = 1, \cdots, n)$$

which fulfil the initial conditions $u_i(x, t)|_{l_i}=0$, where l_i are the lines $x=\mu_i t$. This problem represents a generalisation of the classical Goursat problem. It also includes Cauchy's problem ($\mu_1=\cdots=\mu_n$). Under certain restrictions he finds some necessary and sufficient conditions concerning the coefficients μ_i for its solvability.

M. Zlámal (Brno)

191:

Cerf, G. Sur certaines équations aux dérivées partielles du second ordre à deux variables indépendantes. J. Math. Pures Appl. (9) **37** (1958), 207–223.

Soit une équation aux dérivées partielles à une fonction inconnue z de deux variables indépendantes x, y, possédant deux familles distinctes de caractéristiques.

L'auteur montre que la condition pour qu'une des familles et une seule soit constituée par des caractéristiques du premier ordre est la même que la condition pour que le

système C_1 (de cinq équations de Pfaff à sept variables) définissant les caractéristiques du second ordre de l'autre famille, admette un système dérivé C_1' de quatre équations de classe 6.

Il fait de cette propriété deux applications: l'une est relative à la transformation de Bäcklund à laquelle conduit la recherche des solutions à 2 dimensions d'un système S de deux équations de Pfaff de classe 6; l'autre amène à donner une condition nécessaire et suffisante pour qu'une des familles de caractéristiques du second ordre, la première par exemple, soit intégrable explicitement.

M. Janet (Paris)

192:

Pettineo, Benedetto. Nuova dimostrazione dei teoremi di esistenza per i problemi al contorno regolari relativi alle equazioni lineari a derivate parziali di tipo ellittico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 32–38.

The proof of the existence of a solution of the Dirichlet (and the Neumann) problem for the elliptic partial differential equation

$$(1)\qquad Eu \equiv \sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k}+\sum_{i=1}^{n} b_i\frac{\partial u}{\partial t}+cu = f,$$

for a bounded domain $D\subset E_n$ with "sufficiently" smooth boundary. The coefficients a_{ik}, b_i, c are real-valued functions of position on an open set $S\supset\bar{D}$ such that E is elliptic in S, $\partial a_{ik}/\partial x_i$, b_i, c are Hölder continous in S. By employing a variant of the parametrix analysis of E. E. Levi [Mem. Mat. Fis. Soc. Ital. Sci. (3) **16** (1910), 3–112], the author proceeds to an existence proof which obviates the necessity for Schauder-type estimates or the use of a fundamental solution. Actually, by the use of a suitable "external" problem, it is shown that the Dirichlet problem for Hölder continuous f and continuous boundary values can be reduced to the solution of a pair of integral equations which, in case a uniqueness property is available for the boundary-value problem (e.g., if $c\leqq 0$ in $\bar{D}$), are always solvable.

C. R. DePrima (Pasadena, Calif.)

193:

Danilyuk, I. I. On the mappings corresponding to solutions of elliptic equations. Dokl. Akad. Nauk SSSR **120** (1958), 17–20. (Russian)

Es handelt sich um geometrische Eigenschaften regulärer Lösungen des elliptischen Systems erster Ordnung

$$(1)\qquad \begin{aligned} u_x-v_y &= a(x, y)u+b(x, y)v,\\ u_y+v_x &= c(x, y)u+d(x, y)v \end{aligned}$$

und der elliptischen Gleichung zweiter Ordnung

$$(2)\qquad \begin{aligned} &A_1(x, y)\frac{\partial^2 U}{\partial x^2}+2B_1(x, y)\frac{\partial^2 U}{\partial x\partial y}+C_1(x, y)\frac{\partial^2 U}{\partial y^2}\\ &\qquad+D_1(x, y)\frac{\partial U}{\partial x}+E_1(x, y)\frac{\partial U}{\partial y}+F_1(x, y)U = 0. \end{aligned}$$

Mit $4A = (a-d)+i(c+b)$, $4B = (a+d)+i(c-b)$; $z = x+iy$, $2\partial/\partial\bar{z} = \partial/\partial x+i\partial/\partial y$, $f = u+iv$ wird (1) aequivalent mit

$$(3)\qquad \partial f/\partial\bar{z} = A(x, y)\bar{f}+B(x, y)\bar{z},$$

Sodann wird gezeigt: unter geeigneten Voraussetzungen existiert immer eine Lösung $f=u+iv$ der Gleichung (3), welche in eindeutiger Weise die Abbildung des Inneren einer Kreisscheibe auf gewisse Riemannsche Flächen darstellt. Ferner: sind die Koeffizienten der Gleichung (2) analytische Funktionen von x und y im abgeschlossenen einfach zusammenhängenden Gebiet G und besteht für sie die Möglichkeit einer analytischen Fortsetzung als analytische Funktion von z und ζ $(=\bar{z})$ im Bizylinder $G_z\times G_\zeta$, so gibt es stets eine Lösung $w=f(z, \bar{z})$, durch welche eine Abbildung des Innern des Gebietes G auf gewisse Riemannsche Flächen dargestellt wird.

M. Pinl (Cologne)

194:

Landis, E. M. Relationship between the growth of the solution of an elliptic equation and the number of its sign alternations. Dokl. Akad. Nauk SSSR **123** (1958), 602–605. (Russian)

Consider the second order linear homogeneous equation $Lu=0$ in a domain $D=\{x=(x_1, \cdots, x_n);\, r_1<|x|<r_2\}$ where $r_1>0$, $r_2\leqq 1$. The coefficient of u in Lu is assumed to be $\leqq 0$ and the coefficients in Lu are assumed to be sufficiently smooth. Let N denote the number of subdomains D_0 of D having the following properties: (a) $u>0$ or $u<0$ throughout D_0 and $u=0$ on the boundary of D_0 which lies in D; (b) there exist limit points of D_0 lying on $|x|=r_1$ and limit points lying on $|x|=r_2$. The author proves the following theorem: There exists a constant $C>0$ depending only upon L such that for every solution u, at least one of the following inequalities holds:

$$\text{(i)}\qquad \max_{|x|=r_2}|u(x)|\Big/\max_{|x|=(r_1r_2)^{\frac12}}|u(x)| > (r_2/r_1)^\alpha,$$

$$\text{(ii)}\qquad \max_{|x|=r_1}|u(x)|\Big/\max_{|x|=(r_1r_2)^{\frac12}}|u(x)| > (r_2/r_1)^\alpha,$$

where $\alpha = N^{1/(n-1)}/C$.

A. Friedman (Berkeley, Calif.)

195:

Landis, E. M. Relationship between the growth of the solution of a parabolic equation and the number of its sign alternations. Dokl. Akad. Nauk SSSR **123** (1958), 787–790. (Russian)

The author extends the methods and results of the paper reviewed above to the case of a general linear second order parabolic equation $\partial u/\partial t=Lu$. The domain D is contained in the cylinder $|x|<1$, $0<t<T$, and the solution u is assumed to vanish on the lateral boundary of D. N is defined to be the number of subdomains D_0 having the properties: (a) $u>0$ or $u<0$ throughout D_0 and $u=0$ on the boundary of D_0 which lies in D; (b) limit points of D_0 lie on $t=0$ and on $t=T$. The author shows that there exists $C>0$ depending only upon L such that if $T\geqq 1$ then

$$\max_x|u(x, T)|/\max_x|u(0, x)|<\exp\{-TN^{2/n}/C\}$$

where n is the dimension of x.

A. Friedman (Berkeley, Calif.)

196:

Bernštein, S. N. Some apriori estimates in Dirichlet's generalized problem. Dokl. Akad. Nauk SSSR **124** (1959), 735–738. (Russian)

Consider the elliptic equation

$$(1)\qquad Ar+2Bs+Ct+2Dp+2Eq+Fz=M$$

in a circle C with boundary Γ. Assume that M and the

coefficients of (1) are functions of (x, y) in $C+\Gamma$, and that A, A_x, A_y, B, B_x, B_y, C, C_x, C_y, D, E, F are continuous in $C+\Gamma$ and bounded by K_1. Let K_2 be the modul of ellipticity of (1) in $C+\Gamma$. Let, finally, $\|\ \|$ denote the L_2-norm in C and assume that $\|M\|<\infty$. The author proves: If z is a solution of (1) in C and vanishes on Γ, if $\partial z/\partial \vartheta$ is continuous in $C+\Gamma$ and if $\|p\|+\|q\|<\infty$, then

$$\|r\|^2+\|s\|^2+\|t\|^2+\|p\|^2+\|q\|^2<N(\|z\|^2+\|M\|),$$

where N depends only on K_1, K_2 and on the radius of C. The author remarks that he has proved essentially the same theorem 50 years ago [Math. Ann. **69** (1910), 82–136].

A. Friedman (Berkeley, Calif.)

197:

Hou, Čun′-i. Dirichlet problem for a class of linear elliptic second-order equations with parabolic degeneracy on the boundary of the domain. Sci. Record (N.S.) **2** (1958), 244–249. (Russian)

Let D be a plane domain bounded by a curve $\Gamma=\sigma+AB$ where AB is a segment on $y=0$ and σ lies in $y>0$. Consider the equation

$$\text{(1)} \qquad Lu \equiv y^m\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}+a(x, y)\frac{\partial u}{\partial y}+b(x, y)\frac{\partial u}{\partial x}+c(x, y)u = 0$$

with analytic coefficients in $\bar{D}$. Keldyš [Dokl. Akad. Nauk SSSR **77** (1951), 181–183; MR **13**, 41] proved that if $1<m<2$, $a(x, 0)>0$, then there exists a bounded solution of (1) with prescribed boundary values on σ. In the present paper the author shows that if $1<m<3/2$, $a(x, 0)>0$, and D lies in $y<1$ then for any prescribed continuous function f in $\bar{D}$ there exists a unique solution u of (1) satisfying

$$\lim_{(x,y)\to\Gamma} u(x, y)/w(x, y) = f$$

where

$$w(x, y) = \int_y^1 \exp\int_t^1 a(x, r)r^{-m}drdt.$$

Several extensions are mentioned. The method is based upon that of Tersenov [ibid. **115** (1957), 670–673; MR **19**, 965] who considered the case $m=1$.

A. Friedman (Berkeley, Calif.)

198:

Étienne, J. Fonction de Green de l'opérateur métaharmonique pour les problèmes de Dirichlet ou de Neumann à l'extérieur d'un cercle ou d'une sphère. I, II. Bull. Soc. Roy. Sci. Liège **27** (1958), 142–156, 189–200.

In queste due Note vengono trovate per la prima volta o riottenute in modo semplice e rigoroso diverse interessanti forme, per mezzo di funzioni semplici costruite con le funzioni di Bessel, della soluzione fondamentale dell'equazione metaarmonica $-\Delta u+\kappa^2 u=0$ nel piano e nello spazio e delle relative funzioni di Green per i problemi di Dirichlet e di Neumann all'esterno di un cerchio e di una sfera.

E. Magenes (Genoa)

199:

Gelig, A. H. The stability of the solutions of the Cauchy problem and of the mixed problem for hyperbolic equations. Dokl. Akad. Nauk SSSR **123** (1958), 591–594. (Russian)

Consider the partial differential equation

$$\text{(1)} \quad u_{tt} = a_{ij}u_{x_ix_j}+2a_{i0}u_{x_it}+a_iu_{x_i}+(a_0-\alpha)u_t+(a-\beta)u+f \quad (t \geqq 0,\ -\infty < x_1, \cdots, x_n < +\infty),$$

with the intial conditions

$$u|_{t=0} = \varphi_0(x_1, \cdots, x_n), \quad u_t|_{t=0} = \varphi_1(x_1, \cdots, x_n).$$

The coefficients and f are functions of $t, x_1, \cdots, x_n$ and satisfy the conditions of the theorem of existence and uniqueness of a generalized solution given by S. L. Sobolev [*Nekotorye primeneniya funkcional'nogo analiza v matematičeskoi fizike*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565]. Theorems are stated concerning the behavior of the solutions as $t\to\infty$. If $u=0$ is a solution of (1), then the results imply a type of asymptotic stability of $u=0$ as $t\to\infty$. Extensions are given to systems of equations and to slightly different types of equations and initial conditions.

This is one of the theorems. If φ is any function of t, $x_1, \cdots x_n$, denote by $\|\varphi\|_{L_q}=\sup\int|\varphi|^q dx$, where the integration is made over a sphere in E_n of radius one, and sup is taken over all these spheres. Denote by $\|\phi\|_W$ the sum of the integrals in E_n of $|\varphi|^2$, $|\varphi_t|^2$, $|\varphi_{x_j}|^2$, $j=1, \cdots, n$. Suppose that either $(\partial a_{ij}/\partial t)\xi_i\xi_j\leqq 0$ for $t\geqq 0$, or $|a_{ij}|\leqq A(t)$, $1\leqq i, j\leqq n$, $\int_0^\infty A(t)dt<\infty$. Suppose that $|a_i-\sum_{j=0}^n \partial a_{ji}/\partial x_j|\leqq A(t)$, $0\leqq i\leqq n$, and $\alpha(t, x_1, \cdots, x_n)\geqq 0$, $\beta(x_1, \cdots, x_n)\geqq\beta_0>0$, a constant. Suppose $\|a\|_{L_q}\leqq A_q(t)$ with $\int_0^\infty A_q(t)dt<\infty$, where $q=n$ for $n>2$, $q=2+\gamma$ for $n=2$, $\gamma>0$ arbitrarily small, $q=2$ for $n=1$. Then $\|u\|_W<\varepsilon$ for all $t\geqq 0$ provided all integrals $\int_0^\infty \|f\|_{L_2}dt$, $\int_{E_n}\varphi_1^2dx$, $\int_{E_n}\beta\varphi_0^2dx$, and $\int_{E_n}[a_{ij}|_{t=0}(\varphi_0)_{x_i}(\varphi_0)_{x_j}]dx$ are sufficiently small.

L. Cesari (Baltimore, Md.)

200:

Barancev, R. G. The exact solution of certain boundary value problems for the equation $\psi_{\sigma\sigma}-K(\sigma)\psi_{\theta\theta}=0$ in a hyperbolic strip $\sigma_0\leqq\sigma\leqq\sigma_1$. Vestnik Leningrad. Univ. **13** (1958), no. 19, 19–38. (Russian. English summary)

The present paper is an exposition in detail of the solution reported earlier [Dokl. Akad. Nauk SSSR **114** (1957), 919–922; MR **19**, 865]. Here $K(\sigma)>0$. Data are imposed on the lines $\sigma=\sigma_0$ and $\sigma=\sigma_1$ as well as on a curve $\theta=s(\sigma)$ with end points on the line. The derivative $s'(\sigma)$ is not necessarily continuous at the end points. A critical parameter is the "orientation" $l(\sigma)=[s(\sigma)-\theta_0]/c$, where $c=\int_{\sigma_1}^{\sigma_0}K^{\frac{1}{2}}d\sigma$. When $l(x)\equiv 0$, the solution of the boundary-value problem is expressible in terms of the eigenfunctions of the Sturm-Liouville problem:

$$B_n''(x)+[\lambda_n^2+N(x)]B_n(x)=0, \quad B_n(0)=B_n(1)=0.$$

The author's task is to generalize the method so as to apply when $l(x)\not\equiv 0$. Writing the solution in terms of the functions $z_n(x)=B_n(x)\exp[i\lambda_n l(x)]$, which satisfy a non-self-adjoint equation of the second order, he proves expansion theorems for the two cases $|l'(x)|<l$ and $l(x)\equiv x$. The proofs are based on contour integration in the λ-plane.

R. N. Goss (San Diego, Calif.)

201:

Barancev, R. G. The Goursat problem for an equation of Chaplygin type. Vestnik Leningrad. Univ. **14** (1959), no. 1, 51–56. (Russian. English summary)

The boundary-value problem considered here is to solve the equation

$$(*) \quad (\alpha-\tau)\psi_{\theta\theta}+4\alpha\tau^2(1-\tau)\psi_{\tau\tau}+4\alpha\tau[1+(\beta-1)\tau]\psi_\tau = 0$$

in the disjoint) region $\Omega_a + \Omega_1$ bounded by the two sides of the strip $a<\tau<1$ and the two characteristics of (*) which pass through an arbitrary point O in the strip. Data are given on the characteristics. A change of variables brings (*) into the form $v_{xx}-v_{tt}+N(x)v=0$ and transforms the boundary conditions accordingly, the point O going into the origin of the xt-plane, the characteristics into the lines $t=\pm x$, and $\Omega_a+\Omega_1$ into the sector bounded by these lines. A solution is assumed in the form of a Fourier series in t, and the formulas for the coefficients are regarded as integral transforms of the function $v(x, t)$ with kernels $\frac{1}{2}\cos nt$, $\frac{1}{2}\sin nt$. When the transforms are applied to the boundary-value problem, the coefficients turn out to satisfy ordinary differential equations of the second order, which are solved. Finally the series is transformed into one which is more rapidly convergent.

The simple method to which this problem is amenable is a by-product of the author's research on a more general problem, which has been reported in a series of papers [cf. the preceding review]. *R. N. Goss* (San Diego, Calif.)

202:

Vahaniya, N. N. The Dirichlet problem for the vibrating string. Soobšč. Akad. Nauk Gruzin. SSR **21** (1958), 131–138. (Russian)

The Dirichlet problem is not in general well posed for hyperbolic equations, but certain cases constitute an exception to the rule [cf. D. G. Bourgin and R. Duffin, Bull. Amer. Math. Soc. **45** (1939), 851–858; MR **1**, 120]. In the present paper the author considers the existence of solutions in the generalized sense of the equation $(\partial^2u/\partial x^2)-\lambda^2(\partial^2u/\partial y^2)=0$ in and on the boundary Γ of the unit square $0\leq x\leq 1$, $0\leq y\leq 1$, such that u becomes equal to a given function $f(x, y)$ on Γ. The results, as shown by Bourgin and Duffin, depend on the rationality of λ. The main theorem here is the following: If $|\lambda-(m/n)|>\text{const}/n^r$, where m, n, r are integers, then the above boundary problem has a generalized solution for all functions $g(s)$ whose rth derivative satisfies a Hölder condition with index α $(0<\alpha<1)$ uniformly for $-\infty<s<\infty$. Here $g(s)$ denotes the given function assumed on the boundary after a change of variable which maps the above unit square onto a rectangle, the ratio of the lengths of the sides of which is λ. It is pointed out that the class of functions to which the theorem assigns $g(s)$ is by no means minimal. The results of this paper supplement to some extent those of F. John [Amer. J. Math. **63** (1941), 141–154; MR **2**, 204], who considered classical solutions in general convex regions. *R. N. Goss* (San Diego, Calif.)

203:

Isakova, E. K. The asymptotic behavior of the solution of a second order parabolic partial differential equation with a small parameter in the highest derivative term. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 935–938. (Russian)

Consider the initial value problem

$$(1)\qquad L_\varepsilon u(x, t;\varepsilon) \equiv \varepsilon u_{xx}-u_t+b(x, t)u_x+c(x, t)u = 0,\quad u(x, 0; \varepsilon) = \psi(x),$$

for $\varepsilon>0$ and $(x, t)\in D_\infty(-\infty<x<\infty,\ 0\leq t\leq T)$, where it is assumed that (a) $\psi(x)\equiv 0$ for $x>0$, ψ has n continuous derivatives in $(-\infty, 0]$, $\psi(0-)\neq 0$, and $\psi^{(n)}$ is Lipschitz continuous in $[-\delta, 0]$ for $\delta>0$ sufficiently small; (b) $|\psi^{(k)}(x)|\leq c_1\exp(c_0x^2)$ for $k=0, 1, \cdots, n$; (c) b, b_x, b_t, b_{xx} and c are Lipschitz continuous in D_∞ and b, b_x, c are bounded there. Let $v(x, t)$ denote the solution of the initial value problem $L_0v=0$, $v(x, 0)=\psi(x)$ and let $l(x, t)$ denote the characteristic of L_0 which passes through the point (x, t). The author announces the following theorem: For all $(x, t)\in D_\infty$, $(x, t)\notin l(0, 0)$, $\lim_{\varepsilon\to 0+}u(x, t; \varepsilon)=v(x, t)$, where the convergence is uniform for $|x|\leq R$, $0\leq t\leq T$ outside of some neighborhood of $l(0, 0)$. In the neighborhood of $l(0, 0)$, $u(x, t; \varepsilon) = v(x, t)+u_0(x, t; \varepsilon)+\sum_{k=1}^n u_k(x, t; \varepsilon)+O[\delta^{n+1}(\varepsilon)]$, where $\delta(\varepsilon)=\sqrt{\varepsilon}\log 1/\varepsilon$, $u_k=O[\delta^k(\varepsilon)]$ uniformly for $|x|\leq R$, $0\leq t\leq T$ $(k=1, \cdots, n)$, and u_0 is an "interior boundary layer" term. In particular, $u_0=O(1)$ in any neighborhood which contains points of $l(0, 0)$ and $u_0=O(\sqrt{\varepsilon})$ elsewhere.

The result is proved by working with the equation

$$(2)\qquad L_\varepsilon u(z, y; \varepsilon) \equiv \varepsilon(1+b^2)^{-1/2}u_{yy}-u_z+c(1+b^2)^{-1/2}u = 0,$$

which the author states is obtained from (1) by replacing (x, t) with the variables (y, z), where y is the x-coordinate of the intersection of $l(x, t)$ with $t=0$ and z is arc length along $l(x, t)$ measured from $(y, 0)$ to (x, t). {In the reviewer's opinion, this is incorrect without further restrictions on $b(x, t)$ in (1).} u_k $(k=1, \cdots, n)$ consists of two terms, one given explicitly and the other defined recursively as the solution of a certain initial value problem. {Reviewer's note: In this connection, the statement in the lemma (p. 936) that $v_k^-(z, y; \varepsilon)$ is uniformly $O(\varepsilon^{k/2})$ for $|y|\leq R$ seems to require the additional condition $R\leq\sqrt{\varepsilon}$.} The author's method is based in part on Feller's construction of the fundamental solution of (2) [Math. Ann. **113** (1937), 113–160]. In particular, this is used to show that as $\varepsilon\to 0+$ the solution of (1) at any point (x, t) depends essentially only on the initial data in an interval of length $2\delta(\varepsilon)$ centered at the intersection of $l(x, t)$ with $t=0$.

D. G. Aronson (Minneapolis, Minn.)

204:

Isakova, E. K. Asymptotic behavior of the solution of a parabolic type of differential equation involving a small parameter. Dokl. Akad. Nauk SSSR **119** (1958), 1077–1080. (Russian)

Consider the equation

$$(*)\qquad L_\varepsilon u(x, t; \varepsilon) \equiv \varepsilon\sum_{i,j=1}^n A_{ij}(x, t)\partial^2u/\partial x_i\partial x_j + \sum_{i=1}^n B_i(x, t)\partial u/\partial x_i+C(x, t)u-\partial u/\partial t = 0,$$

for $\varepsilon>0$, $x=(x_1, \cdots, x_n)\in E^n$ and $t\in[0, T]$, where A_{ij}, B_i, C have bounded derivatives up to $2n$th order with respect to x_i, $\partial A_{ij}/\partial t$ and $\partial B_i/\partial t$ are bounded, and $\sum A_{ij}\xi_i\xi_j\geq\alpha\sum\xi_i^2$ for some constant $\alpha>0$ in $E^n\times[0, T]$. The result announced in the paper reviewed above is extended to the initial value problem consisting of (1) and $u(x, 0; \varepsilon)=\Psi(x)$, where $\Psi(x)$ has bounded derivatives of order $2n$ except on some surface $S: F(x)=0$ and the derivatives of Ψ up to order n have jump discontinuities across S. The proof again makes use of an equation, corresponding to (2) in the preceding review, which is obtained from (*) by replacing (x, t) by (y, z). {The reviewer's remark following (2) also applies here.} The study of the behavior of $u(x, t; \varepsilon)$ in the neighborhood of the characteristics of L_0 issuing from S is reduced to the one-dimensional case treated in the previous paper by observing that the second derivative of u normal to S is $O(\varepsilon^{-1})$ while those in directions tangent to S are $O(\varepsilon^{-1/2})$.

Similar methods are applied to investigate the solution of (*) in $0 \leqq x_1 < \infty$, $0 \leqq t \leqq T$ subject to $u(x, 0) = \Psi_1(x)$ and $u(x, t)|_{x_1=0} = \Psi_2(x_2, \cdots, x_n, t)$, where Ψ_1, Ψ_2 have $2n$ bounded derivatives with respect to x, $\Psi_1(0, x_2, \cdots, x_n) = \Psi_2(x_2, \cdots, x_n, 0)$, and $B_1 < 0$. Here the solution $v(x, t)$ of the corresponding problem for L_0 is continuous, but its first derivatives may have jump discontinuities across any characteristic of L_0 issuing from $t = x_1 = 0$. The author gives asymptotic formulas for u and its first derivatives in the neighborhood of such characteristics. The first boundary value problem for (*) in the case $n = 1$ is also discussed. *D. G. Aronson* (Minneapolis, Minn.)

205:

Ventcel′, T. D. Certain quasilinear parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 21–24. (Russian)

Mixed initial and boundary value problems are discussed for the systems:

$$(1)\quad \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{N} b_{ij}(x, t, u_1, \cdots, u_N)\frac{\partial u_j}{\partial x} + \sum_{j=1}^{N} c_{ij}(x, t, u_1, \cdots, u_N)u_j + f_i(x, t) \quad (i = 1 \cdots, N);$$

$$(2)\quad \varepsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t}, \quad 0 = \frac{\partial v}{\partial t} + v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x};$$

and

$$(3)\quad \varepsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{\partial \varphi(v)}{\partial x}, \quad \varepsilon\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x};$$

where $\varphi(v)$ is defined for $v > v_0 \geqq -\infty$, $\varphi'(v) > 0$, $\varphi''(v) \leqq 0$, and $\int^{v_0} (\varphi'(v))^{\frac{1}{2}} dv = -\infty$. Corresponding initial and boundary conditions are:

$$(1^*)\quad u_i(x, 0) = u_i^{(0)}(x), \quad u_i(x_j, t) = u_i^{(j)}(t) \quad (i = 1, \cdots, N;\ j = 1, 2;\ x_1 \leqq x \leqq x_2);$$

or

$$(1^{**})\quad u_i(x, 0) = u_i^{(0)}(x) \quad (-\infty < x < \infty);$$

$$(2^*)\quad u(x, 0) = u_0(x), \quad u(x_1, t) = u(x_2, t) = 0, \quad v(x, 0) = v_0(x);$$

$$(3^*)\quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x).$$

The author shows that under reasonable assumptions these problems admit classical solutions in a rectangle $-\infty \leqq x_1 \leqq x \leqq x_2 \leqq \infty$, $0 \leqq t \leqq T$. He also states conditions under which the solution exists for all positive T. Proofs are omitted. *R. Finn* (Stanford. Calif.)

206:

Yamabe, Hidehiko. A unique continuation theorem of a diffusion equation. Ann. of Math. (2) **69** (1959), 462–466.

Let D be a domain in a C^4 Riemannian manifold with compact closure and smooth boundary ∂D. Let $U(x, t)$ satisfy for $x \in D$ and $t > 0$ the equation $\partial U/\partial t = \Delta U$, where Δ is the Laplace-Beltrami operator, and assume that $U(x, t) = 0$ for $x \in \partial D$ and all t. It is proved that $U(x, t)$ cannot vanish in an open set of D for some $t_0 > 0$ without being identically zero. The proof makes use of a Fourier representation of $U(x, t)$ and the corresponding theorem for elliptic equations due to Aronszajn [C. R. Acad. Sci. Paris **242** (1956), 723–725; MR **17**, 854] and others. *M. Schechter* (New York, N.Y.)

207:

Il′in, A. M.; and Oleĭnik, O. A. Behavior of solutions of the Cauchy problem for certain quasilinear equations for unbounded increase of the time. Dokl. Akad. Nauk SSSR **120** (1958), 25–28. (Russian)

Es handelt sich um das asymptotische Verhalten von Lösungen des Cauchyschen Problems der partiellen Differentialgleichungen (1) $\partial u/\partial t + \partial\varphi(u)/\partial x = \varepsilon\partial^2 u/\partial u^2$, $\varepsilon > 0$ und (2) $\partial u/\partial t + \partial\varphi(u)/\partial x = 0$ für $t \to \infty$. Für $\varphi(u) \equiv u^2/2$ waren diese Fragen zuletzt von E. Hopf behandelt worden [cf. E. Hopf, Comm. Pure Appl. Math. 3 (1950), 201–230; MR **13**, 846]. Verfasser betrachten Lösungen $u(t, x)$ für $t \geqq 0$, welche der Anfangsbedingung (3) $u|_{t=0} = u_0(x)$, $-\infty < x < +\infty$ genügen. $u_0(x)$ ist eine beschränkte meßbare Funktion. $\varphi(u)$ gestattet nach Voraussetzung stetige Ableitungen vierter Ordnung, ferner soll $\varphi''(u) \geqq \mu > 0$ gelten und

$$u_0(x) \to u_-, \text{ wenn } x \to -\infty \text{ und } u_0(x) \to u_+, \text{ wenn } x \to +\infty.$$

Dann wird zunächst die Existenz einer im Bereich $R\{y \geqq 0, -\infty < x < +\infty\}$ beschränkten Funktion $u_\varepsilon(t, x)$ bewiesen, die für $t > 0$ der Gleichung (1) genügt und die vorgeschriebenen Anfangswerte (3) annimmt. Als eine solche Lösung ergibt sich in eindeutiger Weise die Funktion $\tilde{u}_\varepsilon(x - Kt)$, $K = [\varphi(u_+) - \varphi(u_-)]/[u_+ - u_-]$. Für $t \to \infty$ gilt gleichmäßig in x $|\tilde{u}_\varepsilon(x - Kt) - u_\varepsilon(t, x)| \to 0$. Analoge Grenzwertsbeziehungen werden für $u_+ = u_- = q$ aufgestellt. Mit $\varepsilon \to 0$ ergibt sich der Übergang zur Behandlung der Differentialgleichung (2). *M. Pinl* (Cologne)

208:

Bureau, Florent. Problèmes correctement posés pour une équation linéaire aux dérivées partielles ultrahyperbolique. C. R. Acad. Sci. Paris **248** (1959), 1469–1470.

L'A. se propose la question suivante: quelles conditions accessoires peut-on ajouter à une équation linéaire aux dérivées partielles du type hyperbolique non normal (équation ultrahyperbolique) pour que le problème ainsi obtenu possède une et une seule solution?

Il signale, par l'exemple ci-dessous, qu'on peut obtenir des problèmes correctement posés si les données initiales sont portées par certaines variétés linéaires à moins de $n - 1$ dimensions.

Voici l'exemple: posons

$$\Delta_{t,p} = \frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_p^2}, \quad \Delta_{x,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_q^2};$$

le problème $\Delta_{t,p}u = \Delta_{x,q}u$, $u(x; 0) = f(x)$ est correctement posé ($p \geqq 2$, $q \geqq 1$). *S. Mizohata* (Kyoto)

209:

Birman, M. Š. Method of quadratic forms in problems with small parameter in the highest derivatives. Vestnik Leningrad. Univ. **12** (1957), no. 13, 9–12. (Russian. English summary)

Let $\{A_\alpha\}$ be a family of self-adjoint positive definite operators in a Hilbert space H, where α is an element of some partially ordered set with $\alpha \geqq \alpha_0$. Denote by $H_\alpha \subset H$ the closure of the domain of definition of A_α with respect to $(A_\alpha u, u)$. For $u, v \in H_\alpha$ write $(A_\alpha u, v) = [u, v]_\alpha$. The author proves the following theorem. Assume: (1) $[u, u]_\alpha \geqq \gamma^2(u, u)$, where $\gamma^2 > 0$ is a constant independent of α and $u \in H_\alpha$; (2) $H_\alpha \subset H_\beta$ and $[u, u]_\alpha \geqq [u, u]_\beta$ for $\alpha_0 \leqq \beta < \alpha$ and $u \in H_\alpha$; (3) H_α is dense in H_{α_0} and, for $u \in H_\alpha$, $[u, u]_\alpha \to [u, u]_{\alpha_0}$ as $\alpha \to \alpha_0$. Then for every $f \in H$ the solution u_α of

$A_\alpha u=f$ converges in the metric $[u,u]_{\alpha_0}$ to the solution u_0 of $A_{\alpha_0}u=f$. In particular, $A_\alpha^{-1}\to A_{\alpha_0}^{-1}$ strongly in H. The author applies this result to various boundary value problems for elliptic equations involving a small parameter in the highest order term, e.g., $A_\alpha u\equiv -\varepsilon_1\Delta^3u+\varepsilon_2\Delta^2u-\Delta u=f(x)$, $u=\partial u/\partial n=\Delta u=$ on Γ, $\varepsilon_1, \varepsilon_2>0$, where $\alpha=(\varepsilon_1,\varepsilon_2)$, $\alpha_0=(0,0)$, and Γ is the boundary of a bounded region $\Omega\in E^n$. {Reviewer's Note: A closely related result was proved by D. Morgenstern [J. Rational Mech. Anal. **5** (1956), 203–216; MR **17**, 1211].}

D. G. Aronson (Minneapolis, Minn.)

210:

Solonnikov, V. Linear differential equations with a small parameter in the terms of highest order. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 454–457. (Russian)

Let $\Omega\in E^n$ be a bounded region with boundary S and consider the boundary value problem

$$(1)\quad L^\varepsilon u^\varepsilon \equiv \sum_{k=1}^r \varepsilon_k L_{2m_k}u^\varepsilon+\sum_{i=1}^n a_i\partial L_{2m_0}u^\varepsilon/\partial x_i+M_{2m_0}u^\varepsilon=f;$$
$$\partial^j u^\varepsilon/\partial n^j=0 \text{ on } S$$

$(j=0,1,\cdots,m_r-1)$, where $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_r)$, $\varepsilon_k>0$, $m_r>m_{r-1}>\cdots>m_0$, and L_s, M_s are elliptic operators of order s. For example,

$$L_{2m_0}u\equiv(-1)^{m_0}\sum_{|\alpha|,|\beta|=m_0}D^\alpha b_{\alpha,\beta}D^\beta u+\sum_{p=0}^{2m_0-1}\sum_{|\alpha|=p}b_\alpha D^\alpha u,$$

where $\alpha=(s_1,\cdots,s_{|\alpha|})$, s_j is an integer, $1\leqq s_j\leqq n$, $D^\alpha=\partial^{|\alpha|}/\partial x_1\cdots\partial x_{s_{|\alpha|}}$. It is assumed that (1) has a solution in $W_2^{2m_r}(\Omega)$ (the space of functions defined on Ω with $L_2(\Omega)$ derivatives of order $\leqq 2m_r$) and $(M_{2m_0}u,u)_{L_2(\Omega)}\geqq C_0\|u\|^2_{W_2^{m_0}(\Omega)}$ for $u\in W_2^{2m_0}(\Omega)\cap\mathring{W}_2^{m_0}(\Omega)$ and $C_0>0$ sufficiently large. The author announces the following. Theorem 1. There exists a subsequence $\{u^{\varepsilon^k}\}$ which converges weakly in the metric

$$(2)\quad [u,v]=(u,v)_{W_2^{m_0}(\Omega)}+\int_\Omega\sum_{k,i=1}^n\sum_{|\alpha|=m_0}a_ia_k\zeta^2\left(\frac{\partial}{\partial x_i}D^\alpha u\right)\left(\frac{\partial}{\partial x_k}D^\alpha u\right)dx,$$

where $\zeta(x)\in C^{2m_r}(\Omega)$, $\zeta\geqq 0$, $\zeta=0$ on S when $\sum a_i\cos(n,x_i)>0$, to a generalized solution u of

$$(3)\quad Lu\equiv\sum_{i=1}^n a_i\partial L_{2m_0}u/\partial x_i+M_{2m_0}u=f;$$
$$\partial^ju/\partial n^j=0 \text{ on } S\quad (j=0,1,\cdots,m_0).$$

$u\in\mathring{H}^{m_0}(\zeta)$ (the completion of $\mathring{W}_2^{m_0+1}(\Omega)$ with respect to (2)) and is a generalized solution of (3) in the sense that it satisfies the identity obtained by m_0-fold integration by parts of $\int\Phi Lu\,dx=\int f\Phi\,dx$ for arbitrary $\Phi\in\mathring{W}_2^{m_0}(\Omega)$. The proof of this result is based on the estimate

$$\sum_{k=1}^r\varepsilon_k\|u^\varepsilon\|^2_{W_2^{m_k}(\Omega)}+[u^\varepsilon,u^\varepsilon]\leqq C_1\|f\|^2_{L_2(\Omega)}.$$

Similar methods have been used by Ladyženskaya [Vestnik Leningrad Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 104–120; MR **19**, 656]. Since the solvability of (3) is not assumed, a byproduct of theorem 1 is the existence of a generalized solution of (3). Under certain unspecified assumptions on the bahavior of the characteristic of L, the author asserts: Theorem 2: If $v\in\mathring{H}^{m_0}(\zeta)$ is a generalized solution of (3) for $f=0$, then $v=0$. Thus u in theorem 1 is unique and every sequence $\{u^\varepsilon\}$ converges weakly to u.

The author indicates that similar methods can be applied to various other problems involving a small parameter. In particular, the results for the Dirichlet problem for $-\Delta u-\varepsilon u_{tt}+u_t+\Sigma a_iu_{x_i}+au=f$, and the mixed initial-boundary value problem for $u_t-\varepsilon L_2u+\sum a_iu_{x_i}+au=f$ and $\varepsilon L_{2r}u+u_{tt}-L_2u=f$ are sketched.

D. G. Aronson (Minneapolis, Minn.)

211:

Pederson, R. N. On the order of zeros of one-signed solutions of elliptic equations. J. Math. Mech. **8** (1959), 193–196.

Let L be an elliptic operator of order m in n real variables, whose coefficients are defined in a neighborhood N of the origin, and let the coefficients of the ith derivatives belong to c^i. The reviewer proved [J. Math. Mech. **7** (1958), 61–67; MR **20** #174] that if a function u in N satisfies $u\geqq 0$ and $Lu\leqq 0$, and if u vanishes to infinite order at the origin, then $u\equiv 0$ in N. In the present paper the author proves that if a function u in N satisfies $u\geqq 0$, $Lu=0$, and if u vanishes at the origin to order k, then $u\equiv 0$ in N. The number k depends only on the modul of ellipticity of L and on bounds on the coefficients of the mth derivatives of L. The case $m=2$ (with $k=2$) was noted by the reviewer in the above mentioned paper.

A. Friedman (Berkeley, Calif.)

212:

Abolinya, V. È.; and Myškis, A. D. A mixed problem for a linear hyperbolic system on the plane. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 87–104. (Russian. Latvian summary)

In this paper, which is a summary of the dissertation of the first author, sufficient conditions are formulated for the existence and uniqueness of solutions of the following problem: to solve the equation

$$\frac{\partial u_i}{\partial t}-\lambda_i(x,t)\frac{\partial u_i}{\partial x}=\sum_{j=1}^m a_{ij}(x,t)u_j+f_i(x,t)\quad(i=1,\cdots,m),$$

under the initial conditions $u_i(x,0)=g_i(x)$ for $0\leqq x\leqq l$ and the boundary conditions (for $0\leqq t\leqq T$):

$$u_i(0,t)=\sum_{j=k+1}^m\left[\alpha_{ij}(t)u_j(0,t)+\int_0^t\beta_{ij}(t;\tau)u_j(0,\tau)d\tau\right]+h_i(t)\quad(i=1,\cdots,k);$$

$$u_i(l,t)=\sum_{j=1}^k\left[\alpha_{ij}(t)u_j(l,t)+\int_0^t\beta_{ij}(t;\tau)u_j(l,\tau)d\tau\right]+h_i(t)\quad(i=k+1,\cdots,m).$$

The functions $\lambda_i(x,t)$, $a_{ij}(x,t)$, $\alpha_{ij}(t)$, $\beta_{ij}(t;\tau)$ are known, suitably differentiable, functions defined in the lT-rectangle or suitable subsets thereof. The sufficient conditions are imposed upon $f_i(x,t)$, $g_i(x)$ and $h_i(t)$. Existence and uniqueness of the solution in the classical sense are proved by reduction of the problem to a system of Volterra integral equations to which the method of successive approximations is applied. Estimates for the solution and

its derivatives are obtained. When $u(x, t)$ is assumed to lie in a certain function space with norm

$$\|u\| = \sup_{[0,T]} \int_0^l |u|dx + \sup_{[0,l]} \int_0^T |u|dt,$$

a weak, or generalized, solution exists and is unique provided the functions f, g, h are appropriately defined. Certain properties of the weak solution, including continuous dependence on the given data, are deduced.

R. N. Goss (San Diego, Calif.)

213:

Villari, Gaetano. Su un problema al contorno per una classe di sistemi di equazioni alle derivate parziali. Boll. Un. Mat. Ital. (3) **13** (1958), 514–521. (English summary)

Let $U(y)$, $V(x)$ be continuous for $0 \leq y \leq b$, $0 \leq x \leq a$, respectively. Let $F(x, y, u, v)$, $G(x, y, u, v)$ be continuous on the set $D: 0 \leq x \leq a$, $0 \leq y \leq b$, $|u - U(y)| \leq A$, $|v - V(x)| \leq B$; let F [G] be uniformly Lipschitz continuous with respect to u [v]; let $|F|$, $|G| \leq L$ on D. Then the pair of integral equations

$$u(x, y) = U(y) + \int_0^x F(s, y, u(s, y), v(s, y))ds,$$

$$v(x, y) = V(x) + \int_0^y G(x, t, u(x, t), v(x, t))dt$$

has a continuous solution u, v on a rectangle $0 \leq x \leq \min(a, A/L)$, $0 \leq y \leq \min(b, B/L)$. The proof is based on a variant of Tonelli's proof [cf. Sansone, *Equazioni differenziali nel campo reale*, vol. 1, Zanichelli, Bologna, 1948; MR **10**, 193; p. 45] of Peano's existence theorem for ordinary differential equations.

P. Hartman (Baltimore, Md.)

214:

Kajiwara, Joji; and Sibagaki, Wasao. A note on an existence proof about the initial value problem for hyperbolic systems. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. **12** (1958), 22–29.

This note is concerned with a technical refinement of methods used by Hartman and Wintner [Amer. J. Math. **74** (1952), 834–864; MR **14**, 475] and the reviewer [Comm. Pure Appl. Math. **5** (1952), 119–154; MR **14**, 655] in obtaining a C^1 solution of a hyperbolic system of quasi-linear partial differential equations in two independent variables, when the coefficients and the initial data are of class C^1. The methods alluded to are iterative and based on considering a succession of semi-linear problems, each produced by substituting, in place of the dependent variables in the coefficients of the given quasi-linear equation, the solution of the previous problem. The convergence of the iterations is obtained from the fact that the solutions of the successive semi-linear problems have uniformly bounded and equicontinuous first derivatives. The authors here present an improved means of estimating the equicontinuity of the derivatives and thereby gain some additional (but not essential) information concerning Hartman's and Wintner's process and also make it possible to simplify an argument of the reviewer.

A. Douglis (College Park, Md.)

215:

Yuan', Čžao-din [Yuan Chao-din]. On the stability of difference schemes for the solution of parabolic differential equations. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 578–581. (Russian)

In this paper the author develops difference schemes for approximating the solution of the initial-boundary value problem with zero boundary data for second order parabolic equations in a space-time cylinder. He outlines two schemes for the heat equation in the space-time cylinder $Q \times [0, T]$, where Q is a domain in the plane. The author observes that the methods are applicable to problems in higher dimensions and to parabolic operators whose spatial part is a formally self-adjoint operator with coefficients depending both on space and time.

Let

$$u_x = [u(x+h, y) - u(x, y)]/h; \quad u_{\bar{x}} = [u(x, y) - u(x-h, y)]/h.$$

Similarly, let u_y and $y_{\bar{y}}$ denote, respectively, the forward and backward difference quotients of $u(x, y)$ with respect to y. Setting $\Delta_h u = u_{x\bar{x}} + u_{y\bar{y}}$ the author uses the two difference schemes:

$$\text{(A)} \quad u(t, A) = \tau\Delta_h u(t-\tau, A) + u(t-\tau, A); \quad u(t, B) = 0;$$

$$\text{(B)} \quad u(t, A) = [(b + \tau\Delta_h)u(t, A)]/(b+1) + u(t-\tau, A)/(b+1); \quad u(t, B) = 0,$$

where A is in Q_h (net points in Q) and B is in Γ_h (the boundary of Q_h). The author exhibits two procedures for computing the solution of (B) using (A). Estimates for the rapidity of convergence are given. A feature of these schemes is their stability using relatively large time steps.

R. K. Juberg (Minneapolis, Minn.)

216:

Èĭdel'man, S. D. Fundamental matrices of the solutions of general parabolic systems. Dokl. Akad. Nauk SSSR **120** (1958), 980–983. (Russian)

Consider the linear parabolic [in the sense of Petrovskiĭ, Byull. Moskov. Gos. Univ. Sect. A **1** (1938), no. 7] system

$$(1) \quad \frac{\partial^{n_i} u_i}{\partial t^{n_i}} = \sum_{2bk_0 + |k| \leq 2bn_j} A_{ij}^{(k_0 k)}(x, t) \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k u_j(x, t) \equiv P_i\left(t, x; D_x, \frac{\partial}{\partial t}\right)u$$

($i = 1, \cdots, N$; summation over repeated indices), where $x = (x_1, \cdots, x_n)$, $b > 0$ an integer, $2bk_0 < n_j$ ($j = 1, \cdots, N$), $D_x^k = \partial^{|k|}/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$, $|k| = \sum_{s=1}^n k_s$. The author announces the extension of his earlier results [Mat. Sb. N.S. **38** (**80**) (1956), 51–92; MR **17**, 857] on the existence and estimation of the fundamental matrix solution $Z(t, \tau; x, \xi)$ of (1) for (x, t), $(\xi, \tau) \in E^n \times [0, T]$, $t > \tau$ and $n_j = 1$, to the case of unrestricted n_j with a much less restrictive hypothesis on the coefficients of (1). In particular, he assumes that (a) the $A_{ij}^{(k_0 k)}(x, t)$ are continuous functions of t in $E^n \times [0, T]$ and this continuity is uniform when $2bk_0 + |k| = 2bn_j$, and (b) the $A_{ij}^{(k_0 k)}$ are bounded and Hölder continuous in x. If, in addition, $n_j = 1$ and $A_{ij}^{(k_0 k)}$ has $k_0 + |k|$ bounded derivatives with respect to t, x which are Hölder continuous in x, then the transpose of Z as a function of τ, ξ is the fundamental matrix solution of the system adjoint to (1). The author's results [op. cit.] on the analytic continuation of Z in case the $A_{ij}^{(k_0 k)}$ are analytic functions of $z_s = x_s + iy_s$ for some s also extend to (1).

The above results are used to study certain initial value problems for non-linear systems of the form

$$(2) \quad \partial^{n_i} u_i/\partial t^{n_i} = P_i(t, x; D_x, \partial/\partial t)u + F_i(t, x, u, \cdots, \partial^{k_0} D_x^k u_j/\partial t^{k_0}, \cdots) \quad (i = 1, \cdots, N),$$

where $2bk_0 + |k| \leq 2bn_j - 1$.

$$u(x, t) = (u_1, \cdots, u_m) \in L_{p,k(t),m} \quad (1 \leq p < \infty)$$

if

$$\|u(x,t)\|_{L_{p,k(t)},m} = \sum_{j=1}^{m} [\int |u_j(x,t)|^p \exp\{-pk(t)\sum_{s=1}^{n} |x_s|^{2b/2b-1}\}dx]^{1/p} < \infty,$$

and $u(x,t) \in M_{p,m}$ if

$$\|u(x,t)\|_{M_{p,m}} = \int_{t_0}^{t_1} \|u(x,t)\|_{L_{p,k(t)},m}\, dt < \infty,$$

where $k(t) = (c-\varepsilon)r/[(c-\varepsilon)^{2b-1} - r^{2b-1}(t-t_0)]^{1/2b-1}$, $c>0$ a constant which occurs in the estimate for Z, $0<\varepsilon<c$, and $t_0 \leq t \leq t_1 = [(c-\varepsilon')/r]^{2b-1}$ for $\varepsilon<\varepsilon'<c$. The initial value problem considered consists of solving (2) subject to

$$(3) \qquad \lim_{t\to t_0+} \|\partial^{k_0}u_i/\partial t^{k_0} - \varphi_i(x)\delta_{k_0,n_i-1}\|_{L_{p,k(t)}} = 0$$
$$(k_0 = 0, 1, \cdots, n_i-1;\ \varphi_i(x) \in L_{p,k(t_0)}).$$

The author states that if (a) and (b), as well as certain regularity assumptions on the operator F, hold then (2), (3) has a solution such that u_i together with its derivatives of order $k_0+|k|$, $2bk_0+|k| \leq 2bn_i-1$, belong to $M_{p,1}$. The proof of this makes use of the integral equations obtained from (2), (3) by regarding F as a known function of x, t and writing the solution of the resulting inhomogeneous problem in terms of Z. Under certain unspecified additional smoothness conditions on the $A_{ij}^{(k_0k)}$ the same result holds for the problem (2),

$$(3') \qquad \lim_{t\to t_0+} \|\partial^{k_0}u_i/\partial t^{k_0} - \varphi_i^{(k_0)}(x)\|_{L_{p,k(t)}} = 0$$
$$(k_0 = 0, 1, \cdots, n_i-1;\quad i = 1, \cdots, N),$$

where $D^j\varphi_i^{(k_0)} \in L_{p,k(t_0)}$ for $|j| \leq (n_i-k_0-1)2b$.

The question of the uniqueness of solutions of problems (1), (3') and (2), (3) is also considered. For (1), (3') the author asserts that if (a) and (c) $A_{ij}^{(k_0k)}$ has $k_0^*+|k^*|$ bounded derivatives with respect to t, x which are Hölder continuous in x, where $k_0^* = k_0+k_0'$, $k^* = k+k'$,

$$2bk_0'+|k'| = (n_1-n_i)2b, \quad n_1 \geq n_2 \geq \cdots \geq n_N,$$

then there is at most one regular solution of (1), (3') such that $\partial^j u_i/\partial t^j \in M_{p,1}$ $(j=0, 1, \cdots, n_i-1)$ and all the derivatives of u_i with $n_i<n_1$ which appear in (1) are Hölder continuous in the appropriate sense. An analogous result holds for (2), (3) under conditions on F. The analyticity of solutions of (2) is also discussed.

D. G. Aronson (Minneapolis, Minn.)

217:

Sidorov, Yu. V. Cauchy problem for a system of linear partial differential equations with a weight exceeding unity. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 560–563. (Russian)

The author studies solutions of the initial value problem

$$(1) \qquad \left.\frac{\partial^k u_i}{\partial t^k}\right|_{t=0} = \varphi_i^{(k)}(x_1, \cdots, x_n)$$
$$(i = 1, \cdots, p;\quad k = 1, \cdots, n_i-1)$$

for a system of equations

$$(2) \qquad \frac{\partial^{n_i}u_i}{\partial t^{n_i}} = \sum_{(k_s)}\sum_{j=1}^{p} A_{ij}^{(k_0,\cdots,k_n)}(t)\frac{\partial^{k_0+\cdots+k_n}u_j}{\partial t^{k_0}\partial x_1^{k_1}\cdots\partial x_n^{k_n}},$$

where $\sum_{(k_s)}$ denotes summation over all $k_0, \cdots, k_n$ for which $\sum_{s=0}^{n} k_s \leq L$, L a positive integer, $k_0<n_j$; $A_{ij}^{(k_0,\cdots,k_n)}(t)$ and $\varphi_i^{(k)}(x_1, \cdots, x_n)$ are complex functions of real arguments.

Under varying assumptions on the structure of the characteristic equation associated with (2), the author determines conditions under which the problem (1), (2) is correctly set, finds estimates of the order of differentiability of the solution, and gives conditions which insure analyticity of the solution as function of x. Some details were not clear to the reviewer.

R. Finn (Stanford, Calif.)

218:

Gal'pern, S. A. Cauchy problem for general systems of linear partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 640–643. (Russian)

This paper is concerned with the initial value problem for a system of equations

$$\frac{\partial}{\partial t}\left[M\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)U\right] = L\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)U$$

defined throughout space $(t, x_1, \cdots, x_n)$, where M and L are square $N\times N$ matrices of linear differential operators in the "spatial" derivatives. The operators occurring as elements of M and N are assumed to have an order $\leq k$, and to have coefficients depending only on t. The N-vector U of unknown functions is required to satisfy the initial condition $U(t_0, x) = \phi(x)$. After taking Fourier transforms, the equations become $d/dt[M(t,\alpha)v] = L(t,\alpha)v$, and a fundamental system of solutions v^l $(l = 1, 2, \cdots, N)$ is supposed determined, where $V(t_0) = \|v_i^l\|_{t=t_0} = E$, the unit matrix. The author investigates the interrelation between the existence and uniqueness of solutions for the initial value problem of the original equation and the transformed equations. For example, the first theorem asserts: If the matrix $V = \|v_i^l\|$ remains bounded in an arbitrary finite part of the space $(\alpha_1, \cdots, \alpha_n)$ for $t_0 \leq t \leq T$ and has rate of growth not exceeding α^p, $p>0$ as $\alpha\to\infty$, then the formula $u = \int V\Phi \exp i(\alpha, x)dx$, where Φ is the Fourier transform of the initial vector ϕ, gives a solution of the initial value problem, when the ϕ_i possess absolutely square integrable derivatives up to order $\lambda = [n/2]+k+p+1$. [See also S. L. Sobolev, Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 3–50; MR **16**, 1029; M. I. Višik, Mat. Sb. N.S. **39 (81)** (1956), no. 1, 51–148; MR **18**, 215.]

A. N. Milgram (Minneapolis, Minn.)

219:

Višik, M. I.; and Lyusternik, L. A. Asymptotic behavior of the solutions of certain boundary problems with oscillating boundary conditions. Dokl. Akad. Nauk SSSR **120** (1958), 13–16. (Russian)

Asymptotic solutions of boundary value problems for elliptic equations with rapidly oscillating boundary data were considered by the authors in an earlier paper [Dokl. Akad. Nauk SSSR **119** (1958), 636–639; MR **20** #4696]. They showed that for such problems their "second iterative process" goes through and a boundary effect occurs [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5 (77), 3–122; MR **20** #2539]. In the present note they apply their methods to hyperbolic and parabolic equations as well. Specific details are given for the case of two space variables and "locally fixed" boundary data having the form of a simple harmonic multiplied by a function with compact support.

I. The parabolic problem studied is

$$(1)\qquad \frac{\partial u}{\partial t}+L_{2k}u = 0,$$

where L_{2k} is an elliptic operator of order $2k$ involving space derivatives and having sufficiently smooth coefficients, which may depend upon x, y, and t. Let Q be a region of the x, y-plane with smooth boundary Γ and let Ω be the cylinder $Q\times T$, where $T=(-\infty<t<\infty)$, with boundary Γ_1. Coordinates ρ and ϕ are introduced in the neighborhood of Γ, ρ radial and ϕ the parameter for Γ. The boundary condition

$$(2)\qquad \left.\frac{\partial^s u}{\partial n^s}\right|_{\Gamma_1} = A_s(\phi, t)e^{i(\omega t+\gamma\phi)}\quad (s = 0, 1, \cdots, k-1)$$

is imposed, where A_s is a smooth function on Γ_1 with support some neighborhood of Γ_1 plus its interior. An asymptotic solution to (1) and (2) having the form $v(\rho, \phi, t)\exp[i(\omega t+\gamma\phi)]$ is sought. If the boundary data is strongly oscillating in either t or ϕ, then a boundary effect takes place. The asymptotic form in ω or γ of the solution can be computed to any order desired. II. The hyperbolic problem given by $\partial^2u/\partial t^2-L_2u=0$ in Ω, with boundary data on Γ_1 given by

$$(3)\qquad u|_{\Gamma_1} = A(\phi, t)e^{i(\omega t+\gamma\phi)}$$

is considered. III. An elliptic problem

$$L_\varepsilon u \equiv \sum_{s=0}^{2l} \varepsilon^s L_{2k+s}u = 0 \quad\text{in } Q,\quad \left.\frac{\partial^s u}{\partial n^s}\right|_{\Gamma} = A_s(\phi)e^{i\omega\phi}$$

is treated (L_ε becomes an elliptic operator when $\varepsilon\to 0$). IV. Lastly, a hyperbolic problem

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-L_2u = 0\quad\text{in } \Omega$$

and u to satisfy (3) is discussed for small ε. L_2 is a second order elliptic operator.

The theory embraces such boundary layer phenomena as the skin effect and the propagation of surface waves.

N. D. Kazarinoff (Ann Arbor, Mich.)

220:

Rasulov, M. L. The residue method of solving mixed problems and some related expansion formulas. Dokl. Akad. Nauk SSSR **120** (1958), 33–36. (Russian)

An integral representation for a sufficiently smooth solution of a mixed problem for a linear system with piecewise smooth coefficients is given. The problem considered is

$$\frac{\partial u_j^{(i)}}{\partial t} = u_{j+1}^{(i)}\quad (j = 0, \cdots, q-2),$$

$$\frac{\partial u_{q-1}^{(i)}}{\partial t} = \sum_{k\le q-1}^{m+k+l\le p} A_{kl}^{(i)}(x)\frac{\partial^l u_k^{(i)}}{\partial x^l}+f^{(i)}(x, t),$$

$$x\in(a_i, b_i)\quad (i = 1, \cdots, n),$$

with boundary conditions

$$\sum_{i=1}^{n}\sum_{k\le q-1}^{l\le p-1}\left\{\alpha_{kl}^{(i)}\left.\frac{\partial^l u_k^{(i)}}{\partial x^l}\right|_{x=a_i}+\beta_{kl}^{(i)}\left.\frac{\partial^l u_k^{(i)}}{\partial x^l}\right|_{x=b_i}\right\}$$

$$+\sum_{i=1}^{n}\sum_{l=0}^{p-1}\left\{\alpha_{ql}^{(i)}\left.\frac{\partial^{l+1}u_k^{(i)}}{\partial t\partial x^l}\right|_{x=a_i}+\beta_{kl}^{(i)}\left.\frac{\partial^{l+1}u_k^{(i)}}{\partial t\partial_x{}^l}\right|_{x=b_i}\right\} = 0$$

and initial conditions $u_k{}^{(i)}(x, 0) = \phi_k{}^{(i)}(x)$ for $x\in(a_i, b_i)$.

A formula for the expansion of an arbitrary vector-function in a series of solutions for the corresponding spectral problem is also given. Convergence is in the sense of L_2. These results generalize some of the previous work of the author [Mat. Sb. N.S. **30** **(72)** (1952), 509–528; MR **14**, 150].

The method given is applicable to some problems, occurring in applications, for which the spectral problem does not involve a self-adjoint operator, for example,

$$(1)\qquad \frac{\partial^2 u}{\partial t^2}+\delta c^2\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right),\quad \left.\frac{\partial u}{\partial r}\right|_{r=a}+\alpha\left.\frac{\partial u}{\partial t}\right|_{r=a} = 0;$$

$$u(b, t) = 0,\quad \left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \phi_k(r)\quad (k = 0, 1),$$

where c, α, β and δ are constants; and

$$(2)\qquad \frac{\partial u_i}{\partial t} = k_i\left(\frac{\partial^2 u_i}{\partial r^2}+\frac{1}{r}\frac{\partial u_i}{\partial r}\right)+f_i(r, t),\quad r\in(a_{i-1}, a_i);$$

$$\left.\frac{\partial u_i}{\partial r}\right|_{r=a_i}+h_i\{u_{i+1}(a_i, t)-u_i(a_i, t)\} = 0,$$

$$\left.\frac{\partial u_i}{\partial r}\right|_{r=a_{i-1}}+H_i\{u_i(a_{i-1}, t)-u_{i-1}(a_{i-1}, t)\} = 0;$$

$$u_i(r, 0) = \phi_i(r), r\in(a_{i-1}, a_i)\quad (i = 1, 2, 3).$$

The results depend upon the theory of asymptotic solutions of ordinary differential equations containing a large parameter but having no turning points.

N. D. Kazarinoff (Ann Arbor, Mich.)

221:

Agranovič, M. S. On analytic solutions of partial differential equations with constant coefficients. Dokl. Akad. Nauk SSSR **124** (1959), 1183–1186. (Russian)

The author considers the partial differential equation with constant complex coefficients

$$(1)\qquad P(D)u(x) \equiv \sum_{0\le\nu\le m} a_\nu D^\nu u(x)$$

$$\equiv \sum_{0\le\nu_k\le m_k} a_{\nu_1\cdots\nu_n}\frac{\partial^{\nu_1+\cdots+\nu_n}u(x)}{\partial x_1{}^{\nu_1}\cdots\partial x_n{}^{\nu_n}} = f(x),$$

where $(x)=(x_1, \cdots, x_r)$ is a point of a real n-dimensional space. The function $f(x)$ and the solution $u(x)$ are assumed to be expansible in Taylor series

$$(2)\qquad f(x) = \sum b_\mu x^\mu,$$

$$u(x) = \sum c_\mu x^\mu \equiv \sum c_{\mu_1\cdots\mu_n}x_1{}^{\mu_1}\cdots x_n{}^{\mu_n},$$

where the coefficients b_μ, c_μ satisfy specific inequalities. Theorems are stated without proof giving general representations for $u(x)$ of the form

$$(3)\qquad u(x) = (2\pi i)^{-n}\sum_\gamma\sum_{-\gamma\le\nu\le m} d_{\gamma\nu}\int_{T_{\mu+\gamma}} P(s)^{-1}s^{-\gamma-1}e^{x\cdot s}ds,$$

$$d_{\gamma\nu} = a_\nu c_\mu \mu!\quad (\mu = \gamma+\nu \ge 0; \mu! = \mu_1!\cdots\mu_n!);$$

$$(*)\qquad \sum_\nu d_{\gamma\nu} = b_\gamma\gamma!,\quad (\gamma\ge 0),$$

where the T_μ are special contours in the n-dimensional complex space of points $s=(s_1, \cdots, s_n)$. The c_μ appearing (in the $d_{\gamma\nu}$) in (3) are the solutions of the systems (*) and are given explicitly. Consideration is given to special

forms of $P(s)$ for which all the T_μ contours coincide, thereby greatly simplifying (3).

J. F. Heyda (Cincinnati, Ohio)

222:

Kreyszig, Erwin. Coefficient problems in systems of partial differential equations. Arch. Rational Mech. Anal. **1** (1958), 283–294.

Consider the system of two partial differential equations $u_{z_k z_k^*}+c_k(z_k, z_k^*)u=0$ $(k=1, 2)$ for the function $u(z_1, z_1^*, z_2, z_2^*)$, the coefficients $c_k(z_k, z_k^*)$ being entire analytic functions. Bergman and Schiffer [Studies in mathematics and mechanics presented to R. v. Mises, pp. 79–87; Academic Press, New York, 1954; MR **16**, 705] showed that solutions of this system can be obtained through a linear operator applied to a pair of analytic functions of two complex variables. These analytic functions are, essentially, $u(z_1, 0, z_2, 0)$ and $u(z_1, 0, 0, z_2^*)$. Thus, if we develop $u(z_1, z_1^*, z_2, z_2^*)$ into a Taylor series in all its variables and denote the Taylor coefficients by u_{mnpq}, we can determine the solution, its domain of regularity, and the location and nature of its singularities from the coefficient subsets u_{m0p0} and u_{m00q}. In the present paper the analytic functions in two variables

$$u_{nq}(z_1, z_2) = \sum_{m,p=0}^{\infty} u_{mnpq} z_1{}^m z_2{}^p \quad (n, q = 0, 1, \cdots)$$

are introduced, and the problem of how far a knowledge of one single $u_{nq}(z_1, z_2)$ determines the singularity behavior of $u(z_1, z_1^*, z_2, z_2^*)$ is studied. In particular, simple conditions are given which guarantee the singularity of $u_{00}(z_1, z_2)$ at a point where $u_{nq}(z_1, z_2)$ is singular and vice versa.

M. Schiffer (Stanford, Calif.)

223:

Mager, V. Sur une équation n-métaharmonique. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 3–4, 133–138. (Romanian. Russian and French summaries)

Consider

$$(1) \qquad \Delta^n u+\lambda_1\Delta^{n-1}u+\cdots+\lambda_n u = 0$$

with boundary conditions $\Delta^i u=f_i(\alpha)$, $i=0, 1, \cdots n-1$, on the circle of radius R, where α represents the angular coordinate and the f_i are supposed to be uniformly convergent Fourier series. Setting

$$u = A_0(\rho)+\sum (A_m(\rho)\cos m\varphi+B_m(\rho)\sin m\varphi)$$

(ρ, φ polar coordinates), the coefficients A, B satisfy (4): $\Delta^n v+\lambda_1\Delta^{n-1}v+\cdots+\lambda_n v=0$, where

$$\Delta v = d^2v/d\rho^2+\rho^{-1}dv/d\rho-m^2\rho^{-2}v.$$

If η is a root of (7): $\eta^n+\lambda_1\eta^{n-1}+\cdots+\lambda_n=0$, a solution of (4) is found by means of the Bessel equation $\Delta v=\eta v$. M. Ghermanescu [Bul. Şti. Şc. Polit. Timişoara 5 (1934), 152–160] thus solved the problem in the case when (7) has only simple roots. The present paper extends the method to the case when (7) has multiple roots.

J. L. Massera (Montevideo)

224:

Protasov, V. I. On linear partial differential equations of infinite order with constant coefficients. Dokl. Akad. Nauk SSSR **121** (1958), 594–597. (Russian)

For the infinite system of ordinary linear differential equations of mth order with constant coefficients

$$(1) \qquad y_k^{(m)}(x) = \sum_{\nu=0}^{m-1}\sum_{n=0}^{\infty} \alpha_n^{(\nu)} y_{n+k}^{(\nu)}(x)$$

with boundary conditions

$$(2) \qquad y_n^{(\nu)}(0) = c_n^{(\nu)} \quad (n = 0, 1, \cdots; \quad \nu = 0, 1, \cdots, m-1)$$

the author states results analogous to those given by him earlier for the first order case [same Dokl. **105** (1955), 218–221; MR **17**, 847]. These results are then used to establish theorems on the partial differential equation

$$(*) \qquad \partial^m u/\partial x^m = \sum_{\nu=0}^{m-1}\sum_{n=0}^{\infty} \alpha_n^{(\nu)}\, \partial^{\nu+n}u/\partial x^\nu\partial z^n,$$

with boundary conditions

$$(**) \qquad (\partial^\nu u/\partial x^\nu)|_{x=0} = v_\nu(z) \quad (\nu = 0, 1, \cdots, m-1).$$

Solutions are sought in the form

$$u(x, z) = \sum_{n=0}^{\infty} z^n y_n(x)/n!,$$

subject to the requirement that this series be absolutely convergent in some region (and it is the determination of $\{y_n(x)\}$ that leads to system (1), (2)). A typical result: Let

$$a_n = \max_{0\le\nu\le m-1} \{|\alpha_n^{(\nu)}|\}, \quad f(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n.$$

If $f(\zeta)$ is analytic in $|\zeta|<R$, and $v_\nu(z)$ $(\nu=0, 1, \cdots, m-1)$ are entire of first order and type $<R$, then (*), (**) has a solution $u(x, z)$ entire in x, z, unique among analytic functions. For x fixed, u has (in z) a growth not exceeding first order and type $<R$; and for z fixed, is in x not greater than first order and normal type.

I. M. Sheffer (University Park, Pa.)

225:

Rutman, M. A. Order of the exponential growth of the solutions of some sets of simultaneous linear partial differential equations. Dokl. Akad. Nauk SSSR **124** (1959), 764–767. (Russian)

Consider the equation

$$(1) \qquad \frac{\partial^{p_1\cdots p_n}y}{\partial t_1{}^{p_1}\cdots\partial t_n{}^{p_n}} - \sum_{(q_1\cdots q_n)} A_{q_1\cdots q_n}\frac{\partial^{q_1+\cdots+q_n}y}{\partial t_1{}^{q_1}\cdots\partial t_n{}^{q_n}} = x \qquad (0 \le t_1, \cdots, t_n < \infty),$$

where x, y are elements of a linear space C_E contained in a Banach space E, and the A's are linear bounded operators on E continuous and uniformly bounded in the t's. Assume that $q_j\le p_j$, $\sum q_j<\sum p_j$. Together with (1) we consider the Cauchy initial conditions

$$(2) \qquad \partial^m y(t_1, \cdots, t_{j-1}, 0, t_{j+1}, \cdots, t_n)/\partial t_j{}^m = 0 \qquad (j = 1, \cdots, n; \quad m = 0, \cdots, p_j-1).$$

Denote by $C_{E,\alpha}$ the set of elements x in C_E satisfying

$$\sup \|x(t_1, \cdots, t_n)\| \exp[-\alpha(t_1+\cdots+t_n)] < \infty.$$

Definition: For every real number α, $k(\alpha)$ is the inf of the numbers β satisfying the following property: if x belongs to $C_{E,\alpha}$ and y satisfies (1), (2) then y belongs to $C_{E,\beta}$. The author states without proof: There exists α_0 such that $k(\alpha)=\alpha_0$ whenever $\alpha\le\alpha_0$ and $k(\alpha)=\alpha$ whenever $\alpha>\alpha_0$. In the second case, the inf is a minimum. Some examples are given in which α_0 is explicitly calculated.

A. Friedman (Berkeley, Calif.)

POTENTIAL THEORY

See also 147, 438.

226:

Müller, Claus; und Leis, Rolf. Über die Potentialfunktionen von Kurvenbelegungen. Arch. Rational Mech. Anal. **2** (1958), 87–100.

By use of a curvilinear coordinate, the behavior of potentials of simple and double distributions on a curve is studied in this paper. Let $\mathfrak{C}$ be a curve in euclidean 3-dimensional space, $\mathfrak{y}$ be an inner point of $\mathfrak{C}$, and $\mathfrak{h}$ and $\mathfrak{b}$ be the principal normal and binormal vectors, respectively. Let s be the length of $\mathfrak{C}$ measured up to the point $\mathfrak{y}$, and $\rho(\mathfrak{y})=\rho(s)$ be the density at $\mathfrak{y}$. First it is shown that the potential $\int_{\mathfrak{c}} \rho(s)|\mathfrak{x}-\mathfrak{y}(s)|^{-1}ds$ behaves like $-2\rho(\mathfrak{y})\log|\mathfrak{y}-\mathfrak{x}|+O(1)$ as $\mathfrak{x}\to\mathfrak{y}$. Secondly, if $d(\mathfrak{y})=d(s)=\sin\alpha\mathfrak{h}(s)+\cos\alpha\mathfrak{b}(s)$ indicates the direction of the axis of the dipole at $\mathfrak{y}=\mathfrak{y}(s)$, then the potential of the double distribution $\int_{\mathfrak{C}} \rho(s)d(s)|\mathfrak{x}-\mathfrak{y}(s)|^{-1}ds$ behaves like

$$2\rho(\mathfrak{y})(d(\mathfrak{y})(\mathfrak{x}-\mathfrak{y})|\mathfrak{x}-\mathfrak{y}|^{-2})-\rho(\mathfrak{y})\kappa(\mathfrak{y})(\mathfrak{h}\mathfrak{b})\log|\mathfrak{x}-\mathfrak{y}|+O(1)$$

as $\mathfrak{x}\to\mathfrak{y}$, where $\kappa(\mathfrak{y})$ means the curvature at $\mathfrak{y}$. Next, in case $\mathfrak{C}$ is not closed, the behavior at the end-points of $\mathfrak{C}$ is investigated. In the above, the curve and the density are assumed to be sufficiently smooth. In the last paragraph these differentiability conditions are examined.

M. Ohtsuka (Lawrence, Kans.)

227:

★**Arbenz, Kurt. Integralgleichungen für einige Randwertprobleme für Gebiete mit Ecken.** Promotionsarbeit. Eidgenössische Technische Hochschule, Zürich, 1958. 43 pp.

Let G be the finite region bounded by a closed sectionally smooth Jordan curve $\Gamma: z=z(t)$, $a\leq t\leq b$, of bounded rotation and without cusps. J. Radon [S.-B. Akad. Wiss. Wien, Abt. IIa, **128** (1919), 1123–1167] modified the classical integral equations for the Dirichlet and Neumann problems of the logarithmic potential by employing the Stieltjes integral and extended the Fredholm method of solution to regions with corners. Following Radon the author derives for such regions the integral equations for the exterior and interior Dirichlet problem, an analogue of the Lichtenstein-Gershgorin equation for the conformal map $w=f(z)$ of G onto the circle $|w|<1$, and the integral equations of the problem of plane stress and the flexure of thin plates. The Dirichlet problem is set up in the form

$$u(P) = \oint d_t[\varphi_P(t)-\varphi_s(t)]f(t),$$

where $\varphi_p(t)$ denotes the angle which the ray Pt from a point P to the point $z(t)$ makes with a fixed direction; $\varphi_s(t)$ is this angle for $P=z(s)$. Define $\overline{\varphi_s}(t)=\varphi_s(t)$ for $t<s$, $=\varphi_s(-s)$ for $t=s$, $=\varphi_t(s)$ for $t>s$; and $\theta_s(t)=\overline{\varphi_s}(t)-\varphi_s(t)$. Then one obtains for the interior and exterior boundary functions $u_i(s)$ and $u_e(s)$ of $u(P)$

$$u_i(s) = \pi f(s)+\oint d_t\theta_s(t)f(t), \quad u_e(s) = -\pi f(s)+\oint d_t\theta_s(t)f(t).$$

For prescribed $u_i(s)$ and $u_e(s)$ one has thus the desired integral equations for $f(t)$. Similarly, his extension of the equation for conformal mapping, viz. for the function $\sigma(s)=\arg f(z(s))$ is of the form $g(s)=\sigma(s)-1/\pi\oint d_t\theta_s(t)\sigma(t)$. The author is thus led to the study of the eigenvalues and functions of the operator $(\theta f)_s=\oint d_t\theta_s(t)f(t)$, i.e. solutions of the problem $\lambda f=\theta f$. He shows: (1) The Fredholm convergence radius ω of $(\theta f)_s$ is $<\pi$, so that the Fredholm alternative applies for $|\lambda|>\omega$; (2) $\lambda=\pm\pi$ are not eigenvalues; (3) all eigenvalues in $|\lambda|>\omega$ are real and symmetrical with respect to the origin and are contained in $-\pi<\lambda<\pi$; two symmetrical eigenvalues have the same multiplicity; (4) the eigenfunctions pertaining to these λ's are continuous, of bounded variation in $[a, b]$, and periodic.

Finally, the author calculates three numerical examples in which G is a square, by replacing the integral equation by a system of linear equations.

S. E. Warschawski (Minneapolis, Minn.)

228:

Aizenberg, L. A. Pluriharmonic functions. Dokl. Akad. Nauk SSSR **124** (1959), 967–969. (Russian)

The author derives a Poisson-Temliakov integral for biharmonic functions and by means of this he obtains characterizations of biharmonic functions and of their boundary values for domains D with boundaries of the form

$$|w| = r_1(\tau), \quad |z| = r_2(\tau), \quad 0 \leq \tau \leq 1;$$

$$r_1(0) = 0, \quad 0 < r_1'(\tau) \leq \tau^{-1}r_1(\tau), \quad r_1'(1) < \infty,$$

$$r_2(\tau) = \exp\left(-\int_0^\tau \tau(1-\tau)^{-1}d\ln r_1(\tau)\right).$$

With the notation

$$\rho = \tau(r_1(\tau))^{-1}r_1(\tau_0)+(1-\tau)(r_2(\tau))^{-1}r_2(\tau_0)e^{i(t-t_0)},$$

$$u = k\rho e^{iQ_0}, \quad 0 \leq k < 1, \quad m = u+r\rho e^{i\alpha}, \quad r > 0,$$

the validity of the formula

$$f(r_1(\tau_0)ke^{iQ_0}, r_2(\tau_0)ke^{i(Q_0-t_0)}) =$$

$$(2\pi)^{-2}\int_0^{2\pi} dt \int_0^1 d\tau \int_0^{2\pi} F(r_1(\tau)m, r_2(\tau)me^{it})d\alpha$$

everywhere in D for sufficiently small r is a necessary and sufficient condition that $f(w, z)$ is biharmonic in D.

H. Tornehave (Copenhagen)

229:

★**Hodge, W. V. D. Theorie und Anwendungen harmonischer Integrale.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1958. viii+246 pp. DM 15.75.

A translation by Viktor Ziegler of the second edition [University Press, Cambridge, 1952; MR **14**, 500] of the original English work, the first edition of which [University Press, Cambridge, 1941] was reviewed in MR **2**, 296.

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

230:

Leont'ev, A. F. On certain solutions of a linear difference equation with linear coefficients. Mat. Sb. N.S. **45** (87) (1958), 323–332. (Russian)

The difference equation

$$L(y) \equiv \sum_0^n (a_kz+b_k)y(z+h_k) = 0 \quad (h_0 < h_1 < \cdots < h_n)$$

is considered. Solutions are sought which are regular in a half-plane $\Re z>\alpha$ and which have only poles for

singularities. Such solutions are said to be of type Γ by the author. $L(y)=0$ does not have a solution of type Γ if and only if either $a_0=0$ or there is an operator M,

$$M(y) = \sum_0^r \beta_m y(z+p_n),$$

such that $M(L(y))=z\sum_1^s d_j y(z+\mu_j)$. If $L(y)=0$ has solutions of type Γ, then at least one must have one of three possible forms, for example, $\int_{-\infty+id}^{\infty+id}\gamma(t)e^{tz}dt$.

N. D. Kazarinoff (Ann Arbor, Mich.)

231:

Vicente Gonçalves, J. Un problème de limite. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 191–192.

The author shows that under suitable restrictions on the real functions $f(x, y)$, $\phi(y)$ the relation

$$\lim_{y=+0}\sum_{i=1}^{\infty} yf(\phi(y), iy) = \int_0^{\infty} f(\lim_{y=+0}\phi(y), \eta)d\eta$$

is valid.

W. Strodt (New York, N.Y.)

SEQUENCES, SERIES, SUMMABILITY

See also 11, 143, 231, 270.

232:

Moser, Leo. On the series, $\sum 1/p$. Amer. Math. Monthly **65** (1958), 104–105.

The author gives a new simple proof for the divergence of the series $\sum 1/p$.

A. Brauer (Chapel Hill, N.C.)

233:

★Zamansky, Marc. Suites exceptionnelles. Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 Décembre 1957, pp. 148–160. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

Let $g(t)$ be defined and have bounded variation over $0 \leqq t \leqq 1$, let $g(0)=1$, and let g be continuous at $t=0$. A series $u_0+u_1+\cdots$ and its sequence $s_0, s_1, \cdots$ of partial sums are evaluable g to s if

$$T_x(g) = \sum_{k\leqq x} g(k/x)u_k \to s \quad (x\to\infty)$$

over the set of real numbers. The "suites exceptionnelles" of the title are divergent sequences evaluable g. If $\sum u_k$ is evaluable g where $g(t)$ is a polynomial having a zero of multiplicity p at $t=1$, then $u_n=o(n^p)$. If $u_n\to 0$ as $n\to\infty$, then $T_{n+1}(g)-T_n(g)\to 0$ as $n\to\infty$. The remainder of the paper involves the functions g_k for which $g_k(t)=(1+z_k^{-1})(1-t)-z_k^{-1}$ and other well-behaved functions g for which $g(1)\neq 0$. In these cases, series evaluable g must either have particular special forms or be evaluable by some iterate of the relatively weak logarithmic mean transformation by which a sequence $s_1, s_2, \cdots$ is evaluable to s if

$$(\log n)^{-1}\sum_{k=1}^{n} k^{-1}s_k \to s \quad (n\to\infty).$$

R. P. Agnew (Ithaca, N.Y.)

234:

Vicente Gonçalves, J. Généralisation de deux propositions de la théorie des séries réelles. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 311–318.

The identity

$$\sum_{k=1}^{n} u_k = nu_{n+1} + \sum_{k=1}^{n} k(u_k-u_{k+1})$$

implies that the two series $\sum u_k$ and $\sum k(u_k-u_{k+1})$ converges to equal values whenever one of them converges and $nu_n\to 0$ as $n\to\infty$. This and similar results are obtained by methods equivalent to use of the Abel partial summation formula.

R. P. Agnew (Ithaca, N.Y.)

235:

Petersen, G. M. Corrigendum: Sets of consistent summation methods. J. London Math. Soc. **33** (1958), 482.

This is a correction of the proof of theorem 2 of the author's paper in the same J. **32** (1957), 377–379 [MR **19**, 646].

R. P. Agnew (Ithaca, N.Y.)

236:

Srivastava, Pramila. On summability factors. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 182–195.

The author gives sufficient conditions for the summability factor $k(x)$ such that the Stieltjes integral $\int^{\infty} k(x)d\alpha(x)$ is summable $|C, \lambda|$ whenever $\int^{\infty} d\alpha(x)$ is bounded $[C, \lambda]$ (i.e. strongly bounded (C, λ)). It is assumed that $\lambda \geqq 0$. These conditions are: (i) $k(t)$ is continuous; (ii) $\int^{\infty}|k(t)|t^{-1}dt<\infty$; (iii) $\int^{\infty}t^{\lambda}|dk^{(\lambda)}(t)|<\infty$ when λ is an integer, or (iii′) $\int^{\infty}t^{[\lambda]+1}|dk^{([\lambda]+1)}(t)|<\infty$ when λ is not an integer; (iv) $k'(t)$ is a monotonic non-increasing function of t. A similar theorem in which summability $[C, \lambda+1]$ followed from ordinary boundedness (C, λ) was given by Borwein [J. London Math. Soc. **29** (1954), 476–486; MR **16**, 464]. Results concerning absolute and strong Riesz summability for series are deduced from the present result. They extend a result of Pati [Duke Math. J. **21** (1954), 271–283; MR **15**, 950].

J. G. Herriot (Stanford, Calif.)

237:

Kangro, G.; and Baron, S. Summability factors for double series summable by Cesàro's method. Dokl. Akad. Nauk SSSR **124** (1959), 751–753. (Russian)

Die Verfasser stellen verschiedene Konvergenzfaktorensätze für das Cesàro-Verfahren und seine Varianten bei Doppelreihen auf. Für die Indizes wird $0\leqq\gamma$, $\delta\leqq\alpha$, β und α, β ganz vorausgesetzt. Eines der Ergebnisse lautet: Die Zahlen ε_{mn} sind genau dann Konvergenzfaktoren des Typs $(C_r^{\alpha,\beta}, C^{\gamma,\delta})$, wenn folgende Bedingungen erfüllt sind:

$$(1) \qquad \sum_{m,n}(m+1)^{\alpha}(n+1)^{\beta}|\Delta_{mn}^{\alpha+1,\beta+1}\varepsilon_{mn}| < \infty,$$

$$(4) \qquad \varepsilon_{mn} = O[(m+1)^{\gamma-\alpha}(n+1)^{\delta-\beta}],$$

$$\sum_m (m+1)^{\alpha}|\Delta_m^{\alpha+1}\varepsilon_{mn}| = O[(n+1)^{\delta-\beta}],$$

$$\sum_n (n+1)^{\beta}|\Delta_n^{\beta+1}\varepsilon_{mn}| = O[(m+1)^{\gamma-\alpha}].$$

Dieselben Bedingungen gelten auch für den Typ $(C_r^{\alpha\beta}, C_b^{\gamma\delta})$ und den Typ $(C_r^{\alpha\beta}, C_r^{\gamma\delta})$, während etwa beim Typ $(C_b^{\alpha\beta}, C^{\gamma\delta})$ noch die Bedingungen

$$\lim_n (n+1)^{\beta-\delta}\Delta_m^{\alpha+1}\varepsilon_{mn} = 0, \quad \lim_m (m+1)^{\alpha-\gamma}\Delta_n^{\beta+1}\varepsilon_{mn} = 0$$

hinzukommen. Bei weiteren Varianten, zum Beispiel $(C_0^{\alpha\beta}, C^{\gamma\delta})$, muß (4) durch die entsprechende o-Bedingung ersetzt werden.

K. Zeller (Tübingen)

238:

Badalyan, G. V. Some boundary properties of a generalized Taylor series. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 2, 3–29; no. 3, 3–22. (Russian. Armenian summary)

The author introduces methods of summability that depend on numbers $\gamma_n \uparrow \infty$, with $\gamma_0 = 0$, $\sum n/\gamma_n = \infty$, in the same way that the Cesàro and Abel methods depend on the sequence $\{n\}$. Let $\sum a_n$ be a series, S_n its partial sums, and introduce (for typographical convenience in this review) $P(m, n) = \prod_{\nu=m}^{n} (\gamma_\nu - \gamma_{m-1})$. Then put

$$S_n^{(1)} = P(1, n-1) \sum_{k=1}^{n} S_{k-1}/P(2, k),$$

and generally

$$S_n^{(\mu)} = P(\mu, n-1) \sum_{k=\mu}^{n} S_{k-1}^{(\mu-1)}/P(\mu+1, k).$$

Call $\sum a_n$ summable (C_γ, μ) if $S_n^{(\mu)} P(\mu+1, n)/\prod_{\nu=\mu+1}^{n} \gamma_\nu$ approaches a finite limit. (When $\gamma_\nu = \nu$ this is the Cesàro method (C, μ).) The author also introduces

$$\mathring{S}_n^{(1)} = S_n^{(1)} P(2, n) \Big/ \prod_{\nu=2}^{n} \gamma_\nu,$$

$$\mathring{S}_n^{(\mu)} = \left\{ \sum_{k=\mu}^{n} \mathring{S}_{k-1}^{(\mu-1)} \prod_{\nu=\mu}^{k-1} \gamma_\nu / P(\mu+1, k) \right\} P(\mu+1, n) \Big/ \prod_{\nu=\mu+1}^{n} \gamma_\nu.$$

Call $\sum a_n$ summable $(\mathring{C}_\gamma, \mu)$ if $\mathring{S}_n^{(\mu)}$ approaches a finite limit. The author shows that $(\mathring{C}_\gamma, \mu)$ and (C_γ, μ) are regular, and gives order conditions that are necessary for summability. He bases a generalization of Abel summability on his generalized Taylor series [same Izv. **6** (1953), no. 5–6, 1–63; **7** (1954), no. 1, 3–33; MR **16**, 578]. Define functions $\omega_n(x)$ by

$$\omega_n(x) = (2\pi i)^{-1} \prod_{\nu=1}^{n} \gamma_\nu \int_{C_n} x^{-t} \left\{ \prod_{\nu=0}^{n} (t + \gamma_\nu) \right\}^{-1} dt,$$

where C_n surrounds the poles of the integrand; these generalize $(1-x)^n$. The summability method A_γ is defined by $\sum a_n = S\ (A_\gamma)$ if $\sum a_n \omega_n(x) \to S$ as $x \to 0+$. It is regular. If the condition $\gamma_n^n \exp(-\kappa \sum_{\nu=1}^{n} \gamma_\nu^{-1}) \to 0$ is satisfied for some positive κ, then a series summable (C_γ, μ) is summable A_γ to the same sum. The method F_γ is defined by putting

$$f(z) = \sum_{n=0}^{\infty} a_n \prod_{\nu=1}^{n} \gamma_\nu \Big/ \prod_{\nu=1}^{n} (z + \gamma_\nu),$$

supposing the series convergent and taking $f(0+)$ as the sum of $\sum a_n$. Under an auxiliary condition, an A_γ-summable series is F_γ-summable. The author then proves Abelian theorems for series $\phi(x) = \sum a_n \omega_n(x)$; for example, that if $a_n \prod_1^n \gamma_\nu / \prod_0^{n-1} (\omega + \gamma_\nu) \to A$ (with $\omega > 0$) then $x^\omega \phi(x) \to A$ as $x \to 0+$; if $a_n \gamma_n \to A$ then $\phi(x)/\log(1/x) \to A$. He next turns to Tauberian theorems. If $\sum a_n$ is summable $(C_\gamma, 1)$, if $\lim_{n\to\infty} \sum_{\nu=n}^{m(n)} 1/\gamma_\nu = 0$ implies $\sum_{\nu=n}^{m(n)} 1/\alpha_\nu = 0$, and if $a_n > -C/\alpha_n$, then $\sum a_n$ converges. If $\sum a_n$ is summable A_γ and

$$a_n = O\left(\gamma_n^{-1} \exp\left\{ -\delta \sum_{\nu=1}^{n} \gamma_\nu^{-2} \right\} \right),$$

with $\delta > 0$, then $\sum a_n$ converges.

R. P. Boas, Jr. (Evanston, Ill.)

239:

Ogieveckiĭ, I. I. Summation of double series by the methods of Cesàro and Abel in the restricted sense. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 119–125. (Russian)

Verfasser behandelt die restringierte Summierbarkeit von Doppelreihen, die ja zum Beispiel für Fourierreihen von Bedeutung ist. Die Doppelreihe $\sum a_{ij}$ heißt bekanntlich restringiert (C, α, β)-summierbar (wo $\alpha, \beta > -1$), kurz: $(C_\lambda, \alpha, \beta)$-summierbar, wenn die Cesàro-Mittel $\sigma_{mn}(\alpha, \beta)$ einem Grenzwert zustreben für $m, n \to \infty$ unter der Nebenbedingung $\lambda^{-1} \leqq m/n \leqq \lambda$ (wo $\lambda \geqq 1$ beliebig). Entsprechend wird die Variante A_λ des Abelverfahrens erklärt. Theorem 1. Eine (C, α, β)-beschränkte und $(C_\lambda, \alpha, \beta)$-summierbare Reihe ist auch A_λ-summierbar. Mit Hilfe eines Umkehrsatzes von I. E. Ogieveckiĭ [Dokl. Akad. Nauk SSSR **110** (1956), 330–333; MR **18**, 733] folgt daraus Theorem 2: Eine Reihe mit den in Theorem 1 genannten Eigenschaften ist $(C_\lambda, \alpha+\eta, \beta+\delta)$-summierbar für $\eta, \delta > 0$. Weiter erhält man einen Konvexitätssatz (Theorem 3): Aus (C, α, β)-Beschränktheit und $(C_\lambda, \alpha', \beta')$-Summierbarkeit (wo $\alpha' > \alpha$, $\beta' > \beta$) folgt $(C_\lambda, \alpha+\eta, \beta+\delta)$-Summierbarkeit für $\eta, \delta > 0$. In Theorem 1 kann man die Beschränktheitsvoraussetzung abschwächen zu $|\sigma_{mn}(\alpha, \beta)| \leqq C$ für $m, n > M$, $\sigma_{mn} = O(n^{\alpha+1})$, $\sigma_{mn} = O(m^{\beta+1})$ [Theorem 4; vergleiche Timan, ibid. **60** (1948), 1129–1132; MR **10**, 32].

K. Zeller (Tübingen)

240:

Endl, Kurt. Sur une généralisation des procédés de sommation de Hausdorff et la solution d'un problème de moments. C. R. Acad. Sci. Paris **248** (1959), 515–518.

Let, for $\alpha > 0$, $\delta^{(\alpha)}$ be the matrix $\left[(-1)^n \binom{m+\alpha}{m-n} \right]$ and let μ be a diagonal matrix $\{\mu_n\}$. Then the matrix $H^{(\alpha)}(\mu) = \delta^{(\alpha)} \mu \delta^{(\alpha)}$ defines a method of summability which, for $\alpha = 0$, is that of Hausdorff. The main result stated is that $H^{(\alpha)}(\mu)$ is regular if and only if $\mu_n = \int_0^1 x^{n+\alpha} d\chi$, where χ is of bounded variation and $\chi(1) - \chi(+0) = 0$. This again reduces to Hausdorff's familiar result when $\alpha = 0$ [on adding $\chi(0) = 0$].

W. W. Rogosinski (Newcastle-upon-Tyne)

241:

Borwein, D. Theorems on some methods of summability. Quart. J. Math. Oxford Ser. (2) **9** (1958), 310–316.

Der Verf. beweist eine Reihe von Inklusionssätzen zwischen Verfahren A_λ [vgl. Borwein, Proc. Cambridge Philos. Soc. **53** (1957), 318–322; MR **19**, 134], (C^*, α) $(=(C, \alpha)$ für $\alpha > -1$, $=(C, -\alpha)^{-1}$ für $\alpha \leqq -1)$,

$$(C, \alpha, \beta): \frac{1}{\binom{n+\alpha+\beta}{n}} \sum_{\nu=0}^{n} \binom{n-\nu+\alpha-1}{n-\nu} \binom{\nu+\beta}{\nu} s_\nu \qquad (\beta > -1, \alpha+\beta > -1)$$

und Hausdorffverfahren. Wir greifen die folgenden Sätze heraus: Für Hausdorffverfahren H und K gilt $A_\lambda KH \supseteqq A_\lambda K$ falls H regulär ist; für alle reellen α ist $(C^*, \alpha) \simeq (H, \alpha)$ [für $\alpha = -1, -2, \cdots$ wurde dies von Lyra bewiesen, Math. Z. **49** (1944), 538–562; MR **6**, 209]; für reelles α und $\beta > -1$ ist $A_0(C^*, \alpha) \simeq A_\beta(C^*, \alpha+\beta)$.

A. Peyerimhoff (Giessen)

242:

Borwein, D. On multiplication of $(C, -\mu)$-summable series. J. London Math. Soc. **33** (1958), 441–449.

Beweis der folgenden Sätze: Sind $\sum a_n$ und $\sum b_n$ $(C, -\mu)$-summierbar und $(C, -\lambda)$ beschränkt $(\lambda \geqq 1, \lambda > \mu)$, so ist

$\sum_n(\sum_{\nu\leq n} a_{n-\nu}b_\nu)$ $(C, -\lambda+\delta)$-summierbar für jedes $\delta>0$. Sind $\sum a_n$, $\sum b_n$ $|C, -\mu|$-summierbar, so ist $\sum_n(\sum_{\nu\leq n} a_{n-\nu}b_\nu)$ $|C, -\mu|$-summierbar für $\mu\geqq 0$. Zur Definition der $(C, -\mu)$-Verfahren dienen die Verfahren $(C^*, -\mu)$—vgl. das vorangehende Referat. Diese Sätze enthalten frühere Ergebnisse von Hardy [vgl. *Divergent series*, Clarendon Press, Oxford, 1949; MR **11**, 25] und Palmer [Arch. Math. **2** (1950), 258–266; MR **12**, 404].

A. Peyerimhoff (Giessen)

243:

Borwein, D. A logarithmic method of summability. J. London Math. Soc. **33** (1958), 212–220.

Für die Verfahren

$$L:\quad \frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}\ (x\to 1-0)$$

und

$$(A,\lambda):\quad (1-x)\sum_{n=0}^{\infty}\frac{x^n}{\binom{n+\lambda}{n}}\sum_{\nu\leq n}\binom{\nu+\lambda-1}{\nu}s_{n-\nu}$$

$$(\lambda>-1,\ x\to 1-0)$$

gilt: (i) $s_n\to s$ (L) genau dann, wenn $s_{n+1}\to s$ (L); (ii) $L\supset(A,\lambda)$ für $-1<\lambda\leqq 1$; (iii) es gibt $s_n\to s$ (L), aber für kein $\lambda>-1$ ist $s_n\to s$ (A,λ); es gibt $s_n\to s$ (A,λ) für jedes $\lambda>1$, aber $s_n\to s$ (L) gilt nicht; (iv) für reguläre Hausdorffverfahren H_χ gilt $LH_\chi\geqq L$.

A. Peyerimhoff (Giessen)

244:

Borwein, D. On products of sequences. J. London Math. Soc. **33** (1958), 352–357.

Sind $a=\{a_n\}$, $b=\{b_n\}$ Folgen und ist $c=a*b=\{c_n\}=\{\sum_{\nu\leq n}a_\nu b_{n-\nu}\}$ mit $c_n\neq 0$ $(n\geqq 0)$, so wird das verallgemeinerte Nörlundverfahren (N, a, b) erklärt durch

$$s_n' = c_n^{-1}\sum_{\nu\leq n}a_{n-\nu}b_\nu s_\nu$$

(es ist $(N,a,b)=N(q)N(p)^{-1}$ mit $Q_n=\sum_{\nu\leq n}q_\nu=c_n$, $P_n=\sum_{\nu\leq n}p_\nu=b_n$, falls $b_n\neq 0$, $n\geqq 0$). Die Verfahren (N,a,b) enthalten die Euler-Knoppschen- und die arithmetischen Mittelverfahren. (N,a,b) heisst bi-regulär, wenn (N,a,b) und (N,b,a) regulär sind. Der Verf. zeigt, dass aus $s_n\to\sigma$ (N,k,a) und $t_n\to\tau$ (N,l,b) die Beziehung

$$\frac{1}{(a*b)_n}\sum_{\nu\leq n}a_{n-\nu}b_\nu s_\nu t_{n-\nu}\to\sigma\tau\ (N, k*l, a*b)$$

folgt, falls $(N, k*a, l*b)$ bi-regulär ist. Anwendungen des Satzes ergeben Resultate über die Summierbarkeit der Produktreihe von Cesàro-summierbaren und Euler-Knopp-summierbaren Reihen. *A. Peyerimhoff* (Giessen)

245:

Borwein, D. On methods of summability based on integral functions. Proc. Cambridge Philos. Soc. **55** (1959), 23–30.

Nach Vorgabe einer ganzen Funktion $p(z)=\sum p_n z^n$, $p_n\geqq 0$, $\sum_{r=n}^{\infty}p_r>0$ wird ein Limitierungsverfahren P^* in folgender Weise erklärt: $s_n\to l$ (P^*), falls $p_s^*(z)=\sum p_n s_n z^n$ einen Konvergenzradius $\rho(ps)>0$ besitzt und eine analytische Fortsetzung $p_s^*(x)$ auf alle $x>0$ erlaubt für die gilt $(p_s^*(x)/p(x))\to l$ $(x\to\infty)$. Ist $\rho(ps)=\infty$, so wird dieses Verfahren mit P bezeichnet. Der Verf. untersucht Vergleichssätze der Form $Q\subseteqq P$ wo Q durch $q_n=p_n/\mu_n$ erklärt sei. Es zeigt sich, dass $Q\subseteqq P^*$ ist, falls $\rho(ps)>0$ und $\mu_n=\int_0^1 t^n\phi(t)dt$ $(n\geqq N)$ ist mit einem ϕ, das zu einer gewissen Funktionenklasse gehört. Es werden viele Beispeile von derartigen Folgen behandelt: u.a.

$$\mu_n=\prod_{r=1}^{k}\frac{\Gamma(n\alpha_r+\beta_r)}{\Gamma(n\alpha_r+\gamma_r)}\quad(\alpha_r>0,\gamma_r>\beta_r>0);$$

$$\mu_n=\kappa\gamma^{-\alpha n}\Gamma(\alpha n+\beta)\quad(\kappa,\alpha,\beta,\gamma>0).$$

A. Peyerimhoff (Giessen)

APPROXIMATIONS AND EXPANSIONS

See also 111, 272, 420, 421, 423.

246:

Talalyan, A. A. Summing of series of $L_p[a, b]$ space bases, $p>1$ by Cesàro's method. Dokl. Akad. Nauk SSSR **124** (1959), 987–989. (Russian)

Verfasser betrachtet eine normierte Basis $\{\varphi_n(x)\}$ in $L_p(a,b)$ (wo $p>1$). Nach einem früheren Ergebnis [Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 31–68; MR **20** #5998] gibt es zu beliebigem meßbaren, fast überall endlichen $f(x)$ Zahlen $a_n\to 0$, so daß $\sum a_n\varphi_n(x)$ dem Maße nach gegen $f(x)$ konvergiert. Mit Hilfe einer Methode von Men'šov [Mat. Sb. (N.S.) **8** (**50**) (1940), 121–136; MR **2**, 281] folgt daraus Satz 1: Man kann die Basis so umordnen, daß es Zahlen $c_n\to 0$ gibt, für die die Reihe $\sum c_n\varphi_{\nu_n}(x)$ vom Verfahren T gegen $f(x)$ summiert wird (gemeint ist wohl "fast überall"). Dabei ist T ein beliebig vorgegebenes Matrixverfahren, dessen Matrix A (in der Folge-Folge-Form) nachstehende Eigenschaften besitzt: Die Zeilennormen sind beschränkt, die Zeilensummen streben gegen 1, die Zeilenmaxima streben gegen 0. Diese Bedingungen werden zum Beispiel von den Cesàro-Verfahren erfüllt. Verfasser gibt dem Satz sogar eine noch schärfere Form: Man darf abzählbar viele Verfahren des genannten Typs vorgeben und erreicht, daß die Reihe von jedem der Verfahren summiert wird.

K. Zeller (Tübingen)

247:

Yu, Chia-yung. Approximation to functions on the positive real axis by generalized polynomials. Acta Math. Sinica **8** (1958), 190–199. (Chinese. English summary)

We generalize in this paper some results of S. Mandelbrojt ["Séries adhérents, régularisation des suites, applications," Gauthier-Villars, Paris, 1952; MR **14**, 542] and S. Agmon [C. R. Acad. Sci. Paris **228** (1949), 1835–1837; MR **11**, 37].

Theorem 1. Let $F(x)$ be a positive function $(x\geqq 0)$ and $\log F(x)$ be a convex function of $\log x$ $(x>0)$. Let $\{\nu_n\}$ $(n=1,2,\cdots)$ be a sequence of complex numbers such that:

(A) $\quad 0<|\nu_1|<|\nu_2|<\cdots<|\nu_n|<\cdots;$

(B) $\lim_{n\to\infty}(|\nu_{n+K}|-|\nu_n|)>0$, where K is a positive integer;

(C) $\quad \limsup_{n\to\infty}\frac{1}{\log|\nu_n|}\log\frac{1}{|\nu_{n+1}|-|\nu_n|}<\infty;$

(D) $\{\nu_n\}$ lie in a half-strip in the $s=\sigma+it$ plane: $|t|<\alpha$, $\sigma>0$, where α is a positive constant;

$$\text{(E)}\qquad \liminf_{n\to\infty}\frac{n}{|\nu_n|} = D_* > 0.$$

Suppose that (1) $\int^\infty F(e^\sigma)e^{-\sigma/2l}d\sigma=\infty$, where $0<l<D_*$. Under these conditions, we have the following conclusions. (α) If $f(x)$ is a function continuous on $[0,\infty)$, and if $\lim_{x\to\infty}f(x)=0$, then given any $\varepsilon>0$, we can find a polynomial of the form

$$\text{(2)}\qquad P(x) = a_0+a_1x^{\nu_1}+a_2x^{\nu_2}+\cdots+a_nx^{\nu_n},$$

such that for $x\geqq 0$,

$$\left|f(x)-\frac{P(x)}{F(x)}\right| < \varepsilon.$$

(β) If $f(x)$ is a function continuous on $[0,\infty)$, and if $\lim_{x\to\infty}(f(x)/F(x))=0$, then given any $\varepsilon>0$, we can find a polynomial $P(x)$ of the form (2) such that for $x\geqq 0$, $|f(x)-P(x)|<\varepsilon F(x)$.

(γ) If $f(x)\in L_p[0,\infty)$ $(1\leqq p<\infty)$, then given any $\varepsilon>0$, we can find a polynomial $P(x)$ of the form (2) such that

$$\int_0^\infty\left|f(x)-\frac{P(x)}{F(x)}\right|^p dx < \varepsilon.$$

The results of S. Mandelbrojt and S. Agmon on uniqueness of the solution of the moment problem can be generalized as follows. Theorem 2. Let $\{\nu_n\}$ $(n=1,2,\cdots)$ be a strictly increasing sequence of positive numbers verifying the conditions (B), (C) and (E) mentioned above. Let $\{m_n\}$ $(n=0,1,2,\cdots)$ be a sequence of positive numbers. Put $B(\sigma)=\sup_{n\geqq 0}(\nu_n\sigma-\log m_n)$ $(\nu_0=0)$. If (3) $\int^\infty B(\sigma)e^{-\sigma/2l}d\sigma=\infty$, where $0<l<D_*$, then

$$S(0) = 0,\quad \int_0^\infty t^{\nu_n}dS(t) = m_n\quad (n\geqq 0)$$

admits at most a non-decreasing solution $S(t)$.

The conditions (1) and (3) can be replaced by analogous conditions. *Author's summary*

248:

Lebed', G. K. Certain problems of the approximation of functions of one variable by algebraic polynomials. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 239–242. (Russian)

The author gives several direct and inverse theorems of the theory of approximation, mainly in the L^p-norm, of which we can quote only some examples. For a function $f(x)$, $-1\leqq x\leqq 1$, an integer m and $0<\alpha<1$, suppose

$$\text{(1)}\quad \|g(x,h^{-1})^{-\alpha}[f^{(m)}(x(1-h^2)^{\frac12}-h(1-x^2)^{\frac12})]-f^{(m)}(x)\|_{L^p}\leqq M,$$

$0<h<1$, where $g(x,t)=t^{-1}(1-x^2)^{\frac12}+t^{-2}$. Then there exists a sequence of algebraic polynomials $P_n(x)$ of degree not exceeding n such that

$$\text{(2)}\qquad \|g(x,n)^{-m-\alpha}[f(x)-P_n(x)]\|_{L^p}\leqq C(m,p,\alpha)M.$$

For $p=\infty$ this is related to results of Dzyadyk and Freud [see the following reviews]. Conversely, for $p^{-1}<\alpha<1$, $f\in L^p$, relation (2) implies (1), with a constant $C'(m,p,\alpha)$ now appearing in the right-hand side of (1). If $\alpha=1$, the difference of $f^{(m)}$ in (1) is to be replaced by a symmetric second difference.

G. G. Lorentz (Syracuse, N.Y.)

249:

Dzyadyk, V. K. A further strengthening of Jackson's theorem on the approximation of continuous functions by ordinary polynomials. Dokl. Akad. Nauk SSSR **121** (1958), 403–406. (Russian)

The author gives a proof of the theorem announced by him earlier [Izv. Akad. Nauk SSSR, Ser. Mat. **22** (1958), 337–354, M.R. **20** #3406]; see also G. Freud [reviewed below]. *R. Pavley* (Syracuse, N.Y.)

250:

Freud, G. Über die Approximation reeller stetigen Funktionen durch gewöhnliche Polynome. Math. Ann. **137** (1959), 17–25.

Let $\Omega(\delta;f)$ denote the "second modulus of continuity" of the function $f(x)$ defined on $[-1,1]$:

$$\Omega(\delta;f) = \sup|f(x+h)-2f(x)+f(x-h)|,$$

the supremum being taken over all x, h with $0\leqq h\leqq\delta$, $(x-h,x+h)\subset[-1,1]$. The following two theorems are proved. Theorem 1: For each continuous function $f(x)$ there is a sequence $\{P_n(x)\}$ of polynomials $P_n(x)$ of degree $\leqq n$ such that

$$|f(x)-P_n(x)|\leqq C[\Omega(n^{-1}(1-x^2)^{\frac12};f)+\Omega(n^{-2};f)]$$
$$(-1\leqq x\leqq 1;\quad n=1,2,\cdots),$$

where C is a constant independent of x, n and f. Theorem 2: Let $f(x)$ be r-times continuously differentiable in $[-1,1]$. Then there are polynomials $P_n(x)$ of degree $\leqq n$ for which

$$|f(x)-P_n(x)|\leqq$$
$$C(r)n^{-r}((1-x^2)^{\frac12}+n^{-1})^r[\Omega(n^{-1}(1-x^2)^{\frac12};f^{(r)})+\Omega(n^{-2};f^{(r)})]$$
$$(-1\leqq x\leqq 1;\quad n=1,2,\cdots).$$

Walter Gautschi (Washington, D.C.)

251:

Freud, G. Eine Ungleichung für Tschebyscheffsche Approximationspolynome. Acta Sci. Math. Szeged **19** (1958), 162–164.

Let $f_k(x)$, $1\leqq k\leqq n$ denote n functions continuous in $[a,b]$ and such that any "f-polynomial" $\alpha_1f_1+\alpha_2f_2+\cdots+\alpha_nf_n$ with arbitrary coefficients α_k has at most $n-1$ zeros in $[a,b]$. Each function $F(x)$ continuous in $[a,b]$ defines its unique Tchebychev f-polynomial $P(F;x)$ characterized by the least maximum deviation:

$$\max_{a\leqq x\leqq b}|F(x)-P(F;x)| < \max_{a\leqq x\leqq b}\left|F(x)-\sum_1^n\alpha_kf_k(x)\right|$$

provided $\sum_1^n\alpha_kf_k(x)\not\equiv P(F;x)$. The author proves that in $[a,b]$

$$|P(F;x)-P(\Phi;x)|\leqq A\varepsilon$$

if

$$|F(x)-\phi(x)|\leqq\varepsilon,$$

where for fixed $F(x)$ the constant A depends on $\phi(x)$ and the functions $f_k(x)$, but not on ε.

E. Kogbetliantz (New York, N.Y.)

252:

Szász, Otto; and Yeardley, Nelson. The representation of an analytic function by general Laguerre series. Pacific J. Math. **8** (1958), 621–633.

Für komplexe Funktionen $f(z)=f(x+iy)$ wird die

Laguerresche Entwicklung einer Ordnung $\alpha > -1$ erklärt als $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(z)$ mit

$$a_n^{(\alpha)} = \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\right\}^{-1} \int_0^{\infty} e^{-x} x^{\alpha} L_n^{(\alpha)}(x) f(x)dx.$$

Ist $p(b)$ bzw. $\bar{p}(b)$ die durch $y^2 < 4b^2(x+b^2)$ bzw. $y^2 \leqq 4b^2(x+b^2)$ erklärte Punktmenge, so gilt: Eine Funktion $f(z)$ besitzt genau dann für

$$z \in p(d_\alpha), \quad d_\alpha = -\limsup\,(\log |a_n^{(\alpha)}|/2\sqrt{n}), \quad d_\alpha > 0,$$

eine Laguerresche Entwicklung die gegen $f(z)$ konvergiert, wenn $f(z)$ für $z \in p(d_\alpha)$ analytisch ist und wenn für jedes b_α mit $0 \leqq b_\alpha < d_\alpha$ eine Zahl B existiert mit

$$|f(z)| \leqq B \exp\{\tfrac{1}{2}x - |x|^{\frac{1}{2}}[b_\alpha^2 - \tfrac{1}{2}(r-x)^{\frac{1}{2}}]\}$$

für $z \in \bar{p}(b_\alpha)$ $(z = re^{i\theta})$. {Bei der Formel (1.5) muss b_2^{α} durch b_α^2 ersetzt werden.} Hierdurch wird ein Ergebnis von Pollard für $\alpha = 0$ [Ann. of Math (2) **48** (1947), 358–365; MR **8**, 455] verallgemeinert.
A. Peyerimhoff (Giessen)

253:

Korovkin, P. P. An asymptotic property of positive methods of summation of Fourier series and best approximation to functions of class Z_2 by linear positive polynomial operators. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 99–103. (Russian)

Sei

$$u_n(t) = 1/2 + \sum_{k=1}^{n} \rho_k^{(n)} \cos kt \geqq 0;$$

$$L_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)u_n(t)dt.$$

Theorem 1: Folgende Aussagen sind gleichbedeutend:

$$(1) \qquad \lim_{n\to\infty} \frac{L_n(f,x)-f(x)}{L_n(\psi,x)-\psi(x)} = \frac{D_2 f(x)}{D_2\psi(x)};$$

$$(2) \qquad \lim_{n\to\infty} \frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}} = 4.$$

(Dabei ist $D_2 f$ die verallgemeinerte, mit Differenzen gebildete zweite Ableitung.) Daß (2) aus (1) folgt, ergibt sich leicht mit Hilfe der Funktionen $1-\cos kx$. Sei nun Z_2 die Klasse der f mit $|f(x+t)-2f(x)+f(x-t)| \leqq t^2$ sowie

$$a_n = \sup_{f\in Z_2} \max_{-\pi\leqq x\leqq \pi} |L_n(f,x)-f(x)|$$

und $b_n = \inf a_n$, genommen über alle zulässigen u_n. Theorem 2: Es gilt $\lim_n n^2 b_n = \pi^2/2$.
K. Zeller (Tübingen)

254:

Havinson, S. Ya. On uniqueness of functions of best approximation in the metric of the space L_1. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 243–270. (Russian)

Let μ be a measure on a set G (except for (ii) below, μ is positive for each open neighborhood of a point of G), H be a subspace of $L^1(G, \mu)$. Then H has the property (U) with respect to a class of functions ω in L^1 if a polynomial $P = c_1\varphi_1 + \cdots + c_n\varphi_n$, $\varphi_i \in H$, of best approximation to ω (if it exists) is unique. Approximation is measured in the L^1 norm $\|\omega - P\| = \int_G |\omega - P| d\mu$. The functions $\varphi_1, \cdots, \varphi_n$ from L^1 form a Čebyšev system if all non-trivial polynomials P in the φ_i have at most $n-1$ zeros on G. Using his previous results [Dokl. Akad. Nauk SSSR **105** (1955), 1159–1161; MR **17**, 842] the author proves, for example, the following. (i) Let the real continuous $\varphi_1, \cdots, \varphi_n$ on $G = [a, b]$ be such that non-trivial P have nowhere dense sets of zeros. Then the system has the property (U) with respect to the class C of real continuous functions (or, equivalently, with respect to a somewhat larger class $T(\mu)$) for all μ if and only if $\varphi_1, \cdots, \varphi_n$ form a Čebyšev system. (ii) Let μ on $[a, b]$, n be fixed. In order that for each Čebyšev system $\varphi_1, \cdots, \varphi_n$, (U) should hold for C, it is necessary and sufficient that the set of points of increase of μ be either a subinterval of $[a, b]$ or a set of exactly n points. (iii) Let D be a domain of the complex plane, then the subset of $L^1(D, \mu)$ consisting of all meromorphic functions on D has the property (U) with respect to $T(\mu)$.
G. G. Lorentz (Boulder, Colo.)

255:

Brudnyĭ, Yu. A. Approximation by integral functions on the exterior of a segment or on a semi-axis. Dokl. Akad. Nauk SSSR **124** (1959), 739–742. (Russian)

The author develops direct and converse theorems on the degree of approximation, by entire functions of exponential type, to a function given on the exterior of $(-1, 1)$ or on the half-line $x \geqq 0$. He also obtains auxiliary inequalities for the derivatives of entire functions, trigonometric polynomials and polynomials. The first theorem states that if f has r uniformly continuous and bounded derivatives outside $(-1, 1)$ and $f^{(r)}$ has modulus of continuity $\omega(h)$ there, then there exist entire functions G_σ of exponential type σ such that

$$(*)\ |f(x) - G_\sigma(x)| \leqq C\{\sigma^{-1}|x|^{-1}(x^2-1)^{\frac{1}{2}} + \sigma^{-2}\}^r \omega\{\sigma^{-1}|x|^{-1}(x^2-1)^{\frac{1}{2}} + \sigma^{-2}\}.$$

For the half-line, (*) is replaced by

$$(**)\ |f(x) - G_\sigma(x)| \leqq C\{\sigma^{-1}x^{\frac{1}{2}}(x+1)^{-\frac{1}{2}} + \sigma^{-2}\}^r \omega\{\sigma^{-1}x^{\frac{1}{2}}(x+1)^{-\frac{1}{2}} + \sigma^{-2}\}.$$

Theorem 2 gives an inequality for G_σ' when $|G_\sigma(u)| \leqq \varphi(u)$ for all real u and $\varphi(u)$ is bounded, and one for $T_n'(x)$ when $T_n(x)$ is a trigonometric polynomial such that $|T_n(x)| \leqq \varphi(x)$. Theorem 3 gives an estimate for $G_\sigma'(x)$ when $G_\sigma(x)$ is bounded by the right-hand side of (*) outside $(-1, 1)$, and an analogous estimate on a half-line, as well as a similar estimate for the derivative of an ordinary polynomial on $[-1, 1]$. Theorems 4 and 5 estimate the modulus of continuity of f and $f^{(r)}$ if (*) or (**) is satisfied.
R. P. Boas, Jr. (Evanston, Ill.)

FOURIER ANALYSIS

See also 253, 255.

256:

Turán, P. On an inequality. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 3–6.

The main result is: Suppose that $h(x) = \sum_{\nu=0}^{N} b_\nu e^{\nu ix}$, where the ν are integers, and that, for a suitable integer $\Delta \geqq 8$ and for suitable indices

$$(\nu_{-1} =)0 = \nu_0 < \nu_1 < \cdots < \nu_k = N(= \nu_{k+1})$$

with $\nu_{l+1} - \nu_l > \Delta$ $(0 \leqq 1 < k)$, we have

$$(1) \qquad |b_{\nu_j}| \geqq 2 \sum_{\substack{\nu_{j-1}<\nu<\nu_{j+1} \\ \nu\neq\nu_j}} |b_\nu| \quad (0 \leqq j \leqq k);$$

then, for all real b,

$$\int_{-\pi}^{\pi} |h(x)|^2 dx \leqq D \int_{b-d}^{b+d} |h(x)|^2 dx,$$

where $d=\pi\{8/(\Delta+1)\}^{\frac{1}{2}}$ and $D=16(\Delta+1)$. In the special case in which $b_\nu=0$ for $\nu \neq \nu_j$ (when (1) is no restriction) more strongly localized inequalities are known; the sharpest (and best possible) as regards length $2d$ of interval has $d=(\pi+\varepsilon)/\Delta$ and $D=c(\varepsilon)/(2d)$ ($\Delta \geqq 1$; $\varepsilon>0$ and arbitrary) [Ingham, Math. Z. **41** (1936), 367–379]. The author proves his result by applying Parseval's formula to $h(x)$ and to $h(x)H(x)$, where $H(x)=\sum_{j=0}^{\Delta} e^{jix}$. As an application he states a theorem on a class of power series having the unit circle as natural boundary.

A. E. Ingham (Cambridge, England)

257:

Skvorcov, P. G. A necessary condition for strong convergence of Vallée-Poussin sums in Orlicz spaces. Kabardin. Gos. Ped. Inst. Uč. Zap. **12** (1957), 43–53. (Russian)

Consider the Orlicz space L^M of 2π-periodic functions with norm $|f|_M$ generated by a function M which fulfills the condition $M(2u) \leqq AM(u)$. If $f \in L^M$, denote by $s_n(f)$ the nth Fourier sum of f. S. M. Lozinskiĭ [Dokl. Akad. Nauk SSSR **51** (1946), 7–10; MR **8**, 149] has shown that $\liminf M(2u)/M(u)>2$ if $\lim |f-s_n(f)|_M=0$ for every $f \in L^M$. Let $s_{n,m}=(m+1)^{-1}(s_{n-m}+\cdots+s_n)$. The author shows that the same conclusion remains valid if we take de la Vallée-Poussin sums $s_{n,m(n)}$ instead of s_n, provided that $\lim m(n)/n=0$. *V. Pták* (Prague)

258:

Hsiang, Fu Cheng. The Fourier series of functions of bounded *p*th power variation. Michigan Math. J. **6** (1959), 55–58.

Let f be a Lebesgue integrable function of period 2π with Fourier series $\frac{1}{2}a_0+\sum A_n(x)$. Let

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}-s$$

and suppose $\lim_{t\to 0} \varphi(t)=0$. Let $p>0$, $0<\alpha<1$, and $\alpha p<1$. If $A_n(t)=o(n^{-\alpha})$ as $n\to\infty$ and

$$(*) \qquad \limsup_{t\to 0} \Big(\sum_r |\varphi(x_r)-\varphi(x_{r-1})|^p\Big)^{1/p} = O(1),$$

where the sup is taken over all partitions, $t=x_0<x_1<\cdots<x_k=2t$, of $[t, 2t]$, then $\sum A_n(x)$ is summable $(C, -\alpha)$ to s. A similar theorem with a hypothesis equivalent to (*) in the case $p=1$ was established by Bosanquet and Offord [Proc. London Math. Soc. (2) **40** (1936), 273–280].

P. Civin (Eugene, Ore.)

259:

Srivastava, Pramila. The summability factors of a Fourier series and the series conjugate to it. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 196–203.

Let $\sum A_n(x)$ denote the Fourier series of $f(x)$ and $\sum B_n(x)$ the corresponding conjugate series. Let

$$\varphi(t) = (f(x+t)+f(x-t)-2s)/2,$$
$$\psi(t) = (f(x+t)-f(x-t))/2,$$
$$\Phi_\alpha(t) = [\Gamma(\alpha)]^{-1}\textstyle\int_0^t(t-u)^{\alpha-1}\varphi(u)du,$$
$$\varphi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t),$$

with similar definitions for $\Psi_\alpha(t)$ and $\psi_\alpha(t)$. It is known that if $\varphi_\alpha(t)$ is of bounded variation then (*) $\sum A_n(x)\lambda_n$ is summable $|C, \alpha|$, where $\lambda_n=(\log n)^{-1-\varepsilon}$, $\varepsilon>0$ [Cheng, Duke Math. J. **15** (1948), 29–36; MR **9**, 580, 735, for the case $0\leqq\alpha\leqq 1$; Sunouchi, Kōdai Math. Sem. Rep. **1954**, 59–62; MR **16**, 464, for the case $\alpha>1$]. It is also known that if $0\leqq\alpha\leqq 1$, and if

$$\text{(i)}\ \Psi_\alpha(+0) = 0 \quad\text{and}\quad \text{(ii)} \int_0^\pi t^{-\alpha}|d\Psi_\alpha(t)| < \infty$$

hold, then (**) $\sum B_n(x)\lambda_n$ is summable $|C, \alpha|$ where $\lambda_n=(\log n)^{-1-\varepsilon}$, $\varepsilon>0$ [S. J. Wang, Acad. Sinica Science Record **2** (1949), 245–249; MR **11**, 658]. Here this result is extended in two directions. It is shown that if $\alpha>0$ and if (i) and (ii) hold, then (**) is summable $|C, \alpha|$ where $|\Delta\lambda_n|$ is non-increasing and the series $\sum n^{-1}|\lambda_n|$ and $\sum n^{[\alpha]}|\Delta^{[\alpha]+1}\lambda_n|$ are convergent. A similar extension for Fourier series is also possible. Several other results for absolute summability of the series (*) are obtained by starting with various order conditions on $\int_0^t |\varphi(u)|du$ or $\int_0^t |\varphi_\alpha(u)|du$. Analogous results for (**) are also obtained utilizing the corresponding integrals involving $\psi(u)$ and $\psi_\alpha(u)$. *J. G. Herriot* (Stanford, Calif.)

260:

Bredihina, E. A. Fourier series as a device for approximation of almost periodic functions. Dokl. Akad. Nauk SSSR **123** (1958), 219–222. (Russian)

For an almost periodic function $f(x)\sim\sum_{k=-\infty}^{\infty} A_k e^{i\Lambda_k x}$, $\Lambda_0=0$, $\Lambda_k<\Lambda_{k+1}$, $\Lambda_{-k}=-\Lambda_k$, $\lim_{k\to\infty}\Lambda_k=\infty$, the author introduces the notations

$$R_\lambda(f) = \sup_x \Big| f(x) - \sum_{|\Lambda_k|\leqq\lambda} A_k e^{i\Lambda_k x}\Big|,$$

$$N(\lambda) = \sum_{0<\Lambda_k\leqq\lambda} 1, \quad E_\lambda(f) = \inf_F \Big\{\sup_x |f(x)-F(x)|\Big\},$$

where F runs through all integral functions bounded on the real axis and of degree $\leqq\lambda$. The principal theorem states that for every $\mu>\lambda$, the error $R_\lambda(f)$ satisfies the inequality $R_2(f)\leqq\Phi(\lambda,\mu)E_\lambda(f)$, where

$$E_\lambda(f) = 1+4\pi^{-1}+2(N(\mu)-N(\lambda)) + 2\pi^{-1}(\ln(\mu+\lambda)-\ln(\mu-\lambda)).$$

The author applies this result to derive theorems concerning the convergence of the Fourier series. We shall mention the following: If there exist numbers $a>0$, $m\geqq 0$ such that $N(\Lambda_n+a\Lambda_n^m)-N(\Lambda_n)=O(\ln\Lambda_n)$ and $f(x)$ satisfies the conditions $\lim_{\delta\to 0}\omega_f(\delta)\ln\delta=0$, then the Fourier series converges. *H. Tornehave* (Copenhagen)

261:

Džrbašyan, M. M. Uniqueness theorems for Fourier transforms and for infinitely differentiable functions. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 6, 7–24. (Russian. Armenian summary)

The author gives theorems along the lines of Wiener's famous remark that a pair of Fourier transforms cannot both be very small at ∞. [Cf. Hirschman, Amer. J. Math. **72** (1950), 200–213; MR **11**, 350; Jenkins, ibid. **73** (1951), 807–812; MR **13**, 550.] The principal theorem, proved by means of Carleman's theorem, is as follows: let $p(t)$ satisfy $p(0)=p'(0)=0$, $p'(t)\uparrow\infty$; let q be the Young conjugate of p. Let f and F be a pair of L^2 Fourier transforms, let $f(t)e^{p(|t|)}\in L$; then f vanishes identically if

$$\liminf_{R\to\infty}\left\{R^{-1}\int_0^\pi q(R\sin\theta)\sin\theta d\theta+ R^{-2}\int_1^R x\left[\int_1^x u^{-2}\log|F(u)F(-u)|du\right]dx\right\}=-\infty.$$

Several corollaries are shown to include several known results. Another result is: let $f(t)=O(e^{-p(|t|)})$, or $f(t)\in L$ and $f(t)=O(e^{-p(t)})$ as $t\to+\infty$, where $p''(x)\geqq C/x$ for $x\geqq 1$, with $C>1$. If $F(x)=O(e^{-\alpha(|x|)})$, then f vanishes identically if

$$\liminf_{R\to\infty}\left\{R^{-1}q(R)-4\pi^{-1}R^{-2}\int_1^R x\left(\int_1^x t^{-2}\alpha(t)dt\right)dx\right\}=-\infty.$$

Applications are given to conditions under which a class of infinitely differentiable functions contains only 0.

R. P. Boas, Jr. (Evanston, Ill.)

262:

Smolickiĭ, H. L. On a singular integral in the summation theory of a multiple Fourier integral. Vestnik Leningrad. Univ. **13** (1958), no. 7, 125–130. (Russian. English summary)

"In this article we prove the existence of continuous functions for which the singular integral $\int_0^\infty f(t)H(R,t)dt$ is unbounded as $R\to\infty$, where $H(R,t)=(R/t)^{1/2}J_{n-(1/2)}(Rt)$. This kernel arises in the Riesz summation of n-tuple Fourier series." (From the author's summary.)

W. H. J. Fuchs (Ithaca, N.Y.)

263:

Gurariĭ, V. P. On the spectrum of increasing functions. Dokl. Akad. Nauk SSSR **121** (1958), 782–785. (Russian)

The author gives two different definitions, involving Fourier transforms of g, of "the spectrum of g" and shows that there are suitable assumptions in his context under which the two definitions give the same set of real numbers.

M. M. Day (Urbana, Ill.)

264:

Burkill, H. A note on mean values. J. London Math. Soc. **34** (1959), 1–4.

Soit $f(x)$ une fonction réelle de la variable réelle x, continue C_r et presque périodique C_{r-1}, soit $M_r(f)$ sa moyenne de Cesàro d'ordre r, alors il existe un nombre ζ tel que $f(\zeta)=M_r(f)$. Pour le sens des notions utiliseés, voir le travail de l'auteur dans Proc. London Math. Soc. (3) **7** (1957), 481–497 [MR **20** #1167].

J. Favard (Paris)

265:

Ostrow, E. H.; and Stein, E. M. A generalization of lemmas of Marcinkiewicz and Fine with applications to singular integrals. Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 117–135.

The generalization referred to is theorem 1: Let μ be a positive measure on $[0, 2\pi]$ satisfying

$$\tau^{-1}\int_0^\tau d\mu \leqq A \quad (0<\tau<2\pi). \tag{1}$$

Let P be a closed subset of $[0, 2\pi]$ and $D(x)=\rho(x, P)$, the distance from x to P. Then

$$I_\mu(x)=\int_0^{2\pi}\frac{D(x+t)}{t^2}d\mu(t)$$

is finite for almost all $x\in P$. More generally, the authors prove (th. 2) that the same is true of

$$I_\mu^\lambda(x)=\int_0^{2\pi}\frac{D^\lambda(x+t)}{t^{\lambda+1}}d\mu(t)$$

for $\lambda>0$. These results are used in the proofs of the following two theorems. Th. 3: Suppose that $F\in L^2(0, 2\pi)$ and is extended periodically. Let μ be a positive measure satisfying (1). If F' exists in a set E of positive measure, then

$$\int_0^\pi [F(x+t)+F(x-t)-2F(x)]^2\frac{d\mu(t)}{t^3}$$

is finite almost everywhere in E. Th. 4: Under the hypotheses of th. 3,

$$\lim_{\varepsilon\to0}\int_\varepsilon^\pi [F(x+t)+F(x-t)-2F(x)]\frac{d\mu(t)}{t^2} \tag{2}$$

exists (and is finite) for almost all $x\in E$. These results originate in theorems of Marcinkiewicz and Plessner. Theorem 4 is more delicate than theorem 3, for, in general, (2) converges only non-absolutely. In fact, Marcinkiewicz has shown that there exists an F with integrable derivative, for which

$$\int_0^\pi |F(x+t)+F(x-t)-2F(x)|\frac{dt}{t^2}$$

is infinite for almost all x. Nevertheless, as theorem 4 shows, for every partition of $[0, \pi]$ into disjoint intervals Δ_i and for every sequence of choices of $\varepsilon_i=\pm1$, the sum

$$\sum_{i=1}^\infty \varepsilon_i\int_{\Delta_i}[F(x+t)+F(x-t)-2F(x)]\frac{dt}{t^2}$$

is finite for almost all x in $(0, 2\pi)$.

N. J. Fine (Princeton, N.J.)

266:

Korenblyum, B. I. A generalization of Wiener's Tauberian theorem and harmonic analysis of rapidly increasing functions. Trudy Moskov. Mat. Obšč. **7** (1958), 121–148. (Russian)

The author gives proofs of theorems announced earlier [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1 (73), 201–203; MR **18**, 892] and some generalizations. The theorems given here are in a slightly different form than announced.

A. Devinatz (St. Louis, Mo.)

267:

Loo, Ching-tsün. Some properties concerning coefficients of double Fourier series. Acta Math. Sinica **7** (1957), 520–532. (Chinese. English summary)

Let f be a periodic function of period 2π and of bounded variation on $(0, 2\pi)$. Let a_n, b_n be the Fourier coefficients of f and $\rho_n=(a_n^2+b_n^2)^{\frac12}$. Under the usual assumption $f(x)=\frac12 f(x+0)+\frac12 f(x-0)$, a classical theorem of N. Wiener [see A. Zygmund, *Trigonometrical series*, Warszawa-Lwów, 1935, p. 221] asserts that f is continuous on $(0, 2\pi)$, if and only if $(\rho_1+2\rho_2+\cdots+n\rho_n)/n\to0$. The present paper investigates the question of whether Wiener's theorem can be generalized to functions of two variables. Let $F(x, y)$ be a function periodic with period 2π in each variable, and of bounded variation on the square $[0, 2\pi; 0, 2\pi]$ in the sense of Hardy-Krause [cf. J. A. Clarkson and C. R. Adams, Trans. Amer. Math. Soc.

35 (1933), 824–854]. Let a_{mn}, b_{mn}, c_{mn}, d_{mn} be the coefficients of the double Fourier series of F and let $\rho_{mn}=(a_{mn}^2+b_{mn}^2+c_{mn}^2+d_{mn}^2)^{\frac{1}{2}}$. The following result is proved. If F is continuous on the square, then

$$(*)\qquad \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{M} mn\rho_{mn}\to 0$$

as $M\to\infty$, $N\to\infty$. Condition (*) does not imply the continuity of F, but it implies that

$$F(x+h, y+k)-F(x-h, y+k)-F(x+h, y-k) + F(x-h, y-k)\to 0,$$

as $h\to+0$, $k\to+0$. Also an example due to F. Riesz [cf. Zygmund, loc. cit., p. 130] of continuous functions of bounded variation with Fourier coefficients $\neq o(1/n)$ is generalized to functions of two variables.

Ky Fan (Notre Dame, Ind.)

268:

Musielak, J. On absolute convergence of multiple Fourier series. Ann. Polon. Math. **5** (1958), 107–120.

The author proves three theorems which generalize theorems due to S. Bernstein, A. Zygmund and O. Szász [Zygmund, *Trigonometrical series*, Dover, New York, 1955; MR **17**, 361; Chap. 6] concerning single Fourier series to multiple Fourier series and which also generalize theorems due to V. G. Chelidze [Dokl. Akad. Nauk SSSR **54** (1946), 117–120; MR **8**, 376], G. E. Reves and O. Szász [Duke Math. J. **9** (1942), 693–705; MR **4**, 217] and I. E. Zak [Soobšč. Acad. Nauk Gruzin. SSR **12** (1951), 129–133; MR **14**, 42] concerning double Fourier series to multiple Fourier series.

S. Izumi (Sapporo)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 99, 101, 247, 262.

269:

Ramanujan, M. S. Generalized moment problems in function spaces. Math. Student **26** (1958), 105–116.

The paper is, according to the author, "mostly a coverage of Lorentz's work" [Ann. of Math. (2) **51** (1950), 37–55; Pacific J. Math. **1** (1951), 411–429; MR **11**, 442; **13**, 470].

270:

Isaacs, G. L. An extension of a limitation theorem of M. Riesz. J. London Math. Soc. **33** (1958), 406–418.

Complementing results for which references are given, this paper gives necessary conditions for Cesàro and absolute Cesàro evaluability of Laplace-Stieltjes transforms $\int_0^\infty e^{-su}dA(u)$ for cases in which $s\neq 0$ and s has a non-positive real part. Let $A(u)$ have bounded variation over each interval $0\leq u\leq U$, let $A_0(\omega)=A(\omega)$, and let

$$A_k(\omega)=\frac{1}{\Gamma(\omega)}\int_0^\omega(\omega-u)^{k-1}A(u)du$$

when $k>0$. It is not assumed that k is an integer. If $k\geq 0$, if $s=it$ where t is real and $t\neq 0$, and if, as $x\to\infty$, the function $\int_0^x e^{-su}dA(u)$ is limitable (C, k) [or $|C, k|$], then, as $\omega\to\infty$, $A_k(\omega)=O(\omega^{2k+1})$ [or $\omega^{-2k}A_k(\omega)$ is limitable $|C, 0|$]. When the real part of s is negative, a similar but more complicated result is obtained.

R. P. Agnew (Ithaca, N.Y.)

271:

Mainra, V. P. On Poisson's formulae. Bull. Calcutta Math. Soc. **49** (1957), 163–176.

The author proves analogues of the Poisson formulae, which relate to the Fourier sine and cosine transforms, [Titchmarsh, *Theory of Fourier integrals*, Clarendon Press, Oxford, 1937, p. 60] for three other types of transforms, namely the Meijer transform, the Laplace transform, and the $\omega_{\mu,\nu}(x)$ transform with kernel

$$\omega_{\mu,\nu}x = x^{\frac{1}{2}}\int_0^\infty J_\mu(t)J_\nu(x/t)t^{-1}dt.$$

J. Blackman (Syracuse, N.Y.)

272:

Lebedeva, L. P. A method of approximation. Vestnik Leningrad. Univ. **14** (1959), no. 1, 134–139. (Russian. English summary)

Let

$$Q_n(f;x)=\frac{\int_0^{2\pi}(1-a_n\sin^2\frac{1}{2}t)^n f(t+x)dt}{\int_0^{2\pi}(1-a_n\sin^2\frac{1}{2}t)^n dt}$$

where $\{a_n\}$ is an arbitrary sequence of real numbers. If $a_n=1$ $(n>N)$, then $Q_n(f;x)$ coincides with the classical singular integral of de La Vallée Poussin. The author proves: a necessary and sufficient condition that $Q_n(f;x)$ converges uniformly to $f(x)\in C_2$ as $n\to\infty$ is that $\lim_{n\to\infty}(1-a_n)^n=0$. He also finds asymptotic estimates of the constants of Kolmogorov-Nikol'skii:

$$\varepsilon_{Q_n}(KW^{(r)}H^{(\alpha)}, x)=\sup_{f\in KW^{(r)}H^{(\alpha)}}|f(x)-Q_n(f;x)| \quad (0\leqq\alpha\leqq 1;\ r=0, 1, 2, \cdots).$$

The order of his estimates cannot be improved.

A. H. Stroud (Madison, Wis.)

273:

Džrbašyan, M. M. On the theory of integral transformations with Volterra kernels. Dokl. Akad. Nauk SSSR **124** (1959), 22–25. (Russian)

The "Volterra kernel" is

$$\nu(z;\mu)=\int_0^\infty\frac{z^{\mu+t}}{\Gamma(1+\mu+t)}dt.$$

The author gives integral representations and asymptotic properties of $\nu(z;\mu)$ and evaluates Mellin and Laplace transforms involving it. He uses these facts to give an L^2 theory of the transform with kernel $\nu(ixy;\frac{1}{2})$, which formally inverts the one-sided Fourier transform. Specifically, if $g\in L^2(0,\infty)$ then

$$f(x)=(2\pi)^{-\frac{1}{2}}\frac{d}{dx}\int_0^\infty\frac{\nu(ixy;\frac{1}{2})}{iy}g(y)dy$$

belongs to $L^2(-\infty,\infty)$; and

$$(2\pi)^{-\frac{1}{2}}\frac{d}{dx}\int_{-\infty}^\infty f(x)\frac{e^{-ixy}-1}{-ix}dx$$

is equal to $g(y)$ almost everywhere on $(0, \infty)$, and to

$$(*)\qquad (2/\pi)^{\frac{1}{2}} \int_0^\infty g(t) \frac{(-ty)^{-\frac{1}{2}} \log(-y/t)}{\pi^2+\log^2(-y/t)}\, dt$$

almost everywhere on $(-\infty, 0)$. Symmetrically, if

$$f(x) = (2\pi)^{-\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{e^{-ixy}-1}{-iy}\, g(y)dy,$$

then

$$(2\pi)^{-\frac{1}{2}} \frac{d}{dy} \int_{-\infty}^\infty \frac{\nu(ixy;\frac{1}{2})}{ix} f(x)dx$$

is equal to $g(y)$ almost everywhere on $(0, \infty)$, and to (*) almost everywhere on $(-\infty, 0)$.

R. P. Boas, Jr. (Evanston, Ill.)

274:

Suschowk, Dietrich. Die Umkehrung einer Klasse singulärer Integrale. Math. Z. **69** (1958), 363–365.

With $T_k(x)$ denoting the kth Chebychev polynomial, the transform

$$u^{(k)}(t, r) = \int_0^{t-r} [(t-\tau)^2-r^2]^{-\frac{1}{2}} T_k([t-\tau]/r)\cdot f(\tau)d\tau,$$

which is encountered in one of the separation-solutions of the wave equation, admits the inversion

$$f(t) = -\lim_{r\to 0} ru_r^{(0)},$$

$$f(t) = (2^{k-1}(k-1)!)^{-1} d^k \lim_{r\to 0} r^k u^{(k)}/dt^k, \quad k = 1, 2, \cdots.$$

The latter formula is true also for non-integer k.

G. Kuerti (Cleveland, Ohio)

275:

Rooney, P. G. On the inversion of general transformations. Canad. Math. Bull. **2** (1959), 19–24.

The "general transformation" in $L_2(0, \infty)$ may be written in the form

$$\text{(i)}\qquad g(x) = (d/dx) \int_0^\infty k(xy)f(y)dy/y$$

where $\int_0^\infty k(xu)k(yu)u^{-2}du = \min(x, y)$. Writing

$$K(s) = s \int_0^\infty e^{-sx}k(x)dx,$$

it is found that $F(s) = \int_0^\infty K(s/x)g(x)dx/x$ is the Laplace transform of $f(y)$. Thus we may use inversion formulae of the Laplace transform to derive inversion formulae for the transform (i).

The author develops general formulae from the Widder-Post operator, the Phragmén operator and the Laguerre series formula [see G. Doetsch, *Handbuch der Laplace Transformation*, Band I, Birkhäuser, Basel, 1950; MR **13**, 230; Chapter 8].

The particular forms taken by the formula of the Widder-Post type for the sine, cosine and complex Fourier transforms are also given.

J. L. Griffith (Kensington)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 227, 301.

276:

Sobolev, V. V. On the theory of diffusion of radiation. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 5, 39–50. (Russian. Armenian summary)

The author considers the integral equation

$$(*)\qquad B(\tau) = g(\tau)+\int_0^\infty K(|\tau-\tau'|)B(\tau')d\tau',$$

the solution of which has the form

$$B(\tau) = g(\tau)+\int_0^\infty \Gamma(\tau', \tau)g(\tau')d\tau',$$

where

$$\Gamma(\tau, \tau') = K(|\tau-\tau'|)+\int_0^\infty K(|\tau-\tau''|)\Gamma(\tau'',\tau')d\tau''$$

is the resolvent. From the last equation one can obtain

$$\frac{\partial\Gamma}{\partial\tau}+\frac{\partial\Gamma}{\partial\tau'} = \phi(\tau)\phi(\tau'),$$

where $\Gamma(0, \tau) = \phi(\tau)$, and from this it follows that

$$\Gamma(\tau, \tau') = \phi(\tau'-\tau)+\int_0^\tau \phi(\alpha)\phi(\alpha+\tau-\tau')d\alpha.$$

This is a new equation for the resolvent, expressed in terms of a function of a single argument.

The function ϕ is in turn given by

$$\phi(\tau) = K(\tau)+\int_0^\infty K(|\tau-\tau'|)\phi(\tau')d\tau'.$$

When the kernel has the form

$$K(\tau) = \int_a^b A(y)e^{-y\tau}dy,$$

as it does in many physical applications, $\phi(\tau)$ can be found by inversion of a Laplace transform. The author carries out as an example the solution for the diffusion of radiation in a plane layer, in which $K(\tau) = \frac{1}{2}\lambda e^{-\tau}$. He points out that a probabilistic interpretation can be given equation (*) and that numerous physical problems lead to such a model. He also treats parenthetically the case in which the upper limit of the integral in (*) is finite.

R. N. Goss (San Diego, Calif.)

277:

Przeworska-Rolewicz, D. Remarque sur un résultat de Kropatchev. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 727–729.

Kropatchev [Trudy. Kazansk. Aviac. Inst. **30** (1955), 115–133] stated conditions for the solution by successive approximations of the equation

$$u(t) = \lambda \int_L \frac{K[t, \tau, u(\tau)]}{t-\tau}\, d\tau,$$

L denoting closed arcs in the complex plane. The author gives an example showing that the conditions are not sufficient. Additional conditions are stated but no proofs are given.

R. C. MacCamy (Pittsburgh, Pa.)

278:

Bădescu, Radu. Sur une équation intégrale linéaire. Com. Acad. R. P. Romîne 8 (1958), 997–1002. (Romanian. Russian and French summaries)

The author solves the integral equation

$$\Phi(z)-\mu\int_C K(s)\,\frac{\Phi(z)-\Phi(s)}{z-s}\,ds = f(z),$$

where Φ is to be found; C is an arc lying in a region D in which f is analytic except for a single singularity z_0 not on C, and Φ is required to have the same property. The solution is found formally as the difference of two integrals of Cauchy type, and conditions are given for its validity.

R. P. Boas, Jr. (Evanston, Ill.)

279:

Gol'dengeršel', È. I. On the spectrum of a Volterra operator in certain Banach spaces. Dokl. Akad. Nauk SSSR **124** (1959), 1195–1198. (Russian)

Let $K(x, y)$ denote an $n\times n$ matrix-valued function defined and continuous on the region $\theta\leqq y\leqq x<\infty$, where $\theta\geqq 0$ is a real parameter, and let f denote a continuous n-tuple of functions defined on the ray $[\theta, \infty)$. Then the Volterra equation

$$(*)\qquad V_\theta\phi(x)-\lambda\phi(x) = f(x)\quad(\theta\leq x<\infty),$$

where $V_\theta f(x)=\int_\theta^x K(x,y)f(y)dy$, possesses a unique continuous solution ϕ. In order to study the dependence of rate of growth of ϕ on that of f, the author introduces Banach spaces $C_\alpha{}^n(\theta,\infty)$ consisting of the continuous functions f on $[\theta,\infty)$ for which

$$\|f\|_\alpha = \sup_{\theta\leq x}\|f(x)\|e^{-\alpha x}<\infty$$

($\|f(x)\|$ and $\|K(x,y)\|$ denote ordinary L_2 norms). A necessary and sufficient condition that V_θ be bounded is

$$\sup_x\int_\theta^x\|K(x,y)\|e^{-\alpha(x-y)}dy<\infty;$$

but V_θ is always closed, so that, if $R_\lambda(V_\theta)=(V_\theta-\lambda I)^{-1}$ denotes the resolvent, then $R_\lambda(V_\theta)C_\alpha{}^n(\theta,\infty)\subset C_\alpha{}^n(\theta,\infty)$ is a necessary and sufficient condition that $\lambda\notin\sigma_\alpha(V_\theta)$, the spectrum of V_θ on the space $C_\alpha{}^n(\theta,\infty)$. Thus, relations between the rates of growth of f and ϕ in (*) may be explored in terms of the dependence of $\sigma_\alpha(V_\theta)$ on the parameters α and θ. Conditions bearing on V_0 are given insuring that

$$\bigcap_{-\infty<\alpha<\infty}\sigma_\alpha(V_\theta)=\{0\},$$

as well as that $\sigma_\alpha(V_\theta)$ should be independent of θ for $\theta\geqq 0$. The estimate

$$r_\alpha(V_0)\leqq\lim_{\theta\to\infty}\sup_{\theta\leq x}\int_\theta^x\|K(x,y)\|e^{-\alpha(x-y)}dy$$

is obtained for the spectral radius of V_0. In the special case $K(x,y)=K(x-y)$, where $\int_0^\infty\|K(t)\|e^{-\alpha t}dt<\infty$, the Paley-Wiener theorem is invoked to give an explicit characterization of $\sigma_\alpha(V_0)$ in terms of the Laplace transform of K. The perturbation of such a Volterra operator by another is studied and a characterization of $\sigma_\alpha(QV_0)$ in terms of $\sigma_\alpha(V_0)$ is given, where Q denotes a constant matrix. The author considers briefly the functional calculus for such operators, the use of Banach spaces with integral norms and the possibility of weakening the continuity requirements on the kernel K. Extensions of the various results to multi-dimensional situations are indicated. The note concludes with a discussion of applications to various boundary value problems. In particular, a condition is derived insuring the boundedness of the solution of a heat flow problem with continuously distributed source term. There are no proofs.

A. Brown (Houston, Tex.)

280:

Manžeron, D. [Mangeron, D.] The solution of some eigenvalue problems transformed into a class of integral equations. Bul. Inst. Politehn. Iaşi (N.S.) 4 (8) (1958), 65–68. (Russian. English and Romanian summaries)

Considering a Fredholm integral equation with a kernel which is a certain symmetric homogeneous polynomial, the author calculates the eigenvalues and eigenfunctions.

A. Friedman (Berkeley, Calif.)

281:

Doležal, Václav. Systems of ordinary linear integro-differential equations. Apl. Mat. 4 (1959), 1–17. (Czech. Russian and English summaries)

The author considers systems with real constant coefficients of the type occurring in the analysis of lumped parameter electrical circuits,

$$(1)\qquad\sum_{k=1}^r\left(\alpha_{ik}x_k+\beta_{ik}\dot{x}_k+\gamma_{ik}\int_0^t x_k d\tau\right)+\delta_i=f_i(t),\qquad i=1,2,\cdots,r.$$

The set of functions $f(t)$ defined on $[0,\infty)$ is said to belong to S if for each $t\in(0,\infty)$ the Lebesgue integral $\int_0^t f(\tau)d\tau$ exists and there are for each function f numbers ξ, $\eta\geqq 0$ for which $|\int_0^t f(\tau)d\tau|\leqq\eta e^{\xi t}$. If in (1) the $f_i\in S$, then $x(t)=(x_1(t),\cdots,x_r(t))\in S$ is called a generalized solution of (1) with initial conditions $c=(c_1,\cdots,c_r)=x(0)$ if $x(t)$ satisfies the system

$$(2)\qquad\sum_{k=1}^r\left(\alpha_{ik}\int_0^t x_k d\tau+\beta_{ik}x_k-\beta_{ik}c_k+\gamma_{ik}\int_0^t\int_0^\tau x_k d\sigma d\tau\right)+\delta_i t = \int_0^t f_i d\tau,\quad i=1,2,\cdots,r$$

almost everywhere. If, more particularly, $x(t)$ is continuous in $[0,\infty)$ ($x(0)=c$), has a continuous derivative in $(0,\infty)$, and satisfies (1) identically in $(0,\infty)$, then $x(t)$ is identified as a classical solution of (1). Existence theorems are stated and proved giving sufficient conditions on the f_i and the coefficients of (1) to insure a unique generalized and/or classical solution of (1), and a matrix formulation for $X(p)$, the Laplace transform of $x(t)$, is obtained. Finally, it is shown that the generalized solution can be obtained as a limit of classical solutions of systems (1) with the same coefficients under specified conditions on these coefficients and a specific convergence criterion. Physical interpretations of the results are given, along with a numerical example.

J. F. Heyda (Cincinnati, Ohio)

FUNCTIONAL ANALYSIS

See also 75, 92, 102, 107, 209, 269, 420, 589.

282:

Ghelfand, I. M. [Gel'fand, I. M.] On some problems of functional analysis. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **11** (1957), no. 3 (22), 58–67. (Romanian)

Translation of the paper in Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6 (72), 3–12 [MR **19**, 293].

283:

★**Sebastião e Silva, J. Conceitos de função diferenciável em espaços localmente convexos. [Concepts of a differentiable function in locally convex spaces.]** Publicações do Centro de Estudos Matemáticos de Lisboa, Instituto de Alta Cultura, Lisbon, 1957. 65 pp. (multigraphed)

This is a very readable self-contained account of differentiation and integration, and of analytic functions, in locally convex linear topological spaces. In part, the monograph consists of detailed proofs of results previously announced [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 743–750; **21** (1956), 40–46; MR **19**, 561]. There is also some new material.

Let X and Y be locally convex spaces, f a function with domain in X and range in Y. Let S be a class of bounded subsets of X containing all one-point subsets and, with a set, also its absolutely convex envelope. Then f is differentiable at a with respect to S in the wide sense if there is a linear map A of X into Y, bounded on each $B\in S$, such that $t^{-1}\{f(a+th)-f(a)\}\to Ah$ as $t\to 0$, uniformly on each $B\in S$. This generalises both differentiability with respect to S, as previously defined [loc. cit.], and also (for suitable S) differentiability in the sense of Fréchet-Hyers. Some, although not all, of the results announced as valid for differentiability with respect to S remain true with this extended definition.

The main novelty in the present treatment is the recognition that two 'dual' theories are desirable in locally convex spaces. For example, local boundedness can be generalised in two ways: (i) f is locally quasi-bounded at a if, for each continuous semi-norm p on Y, there is a neighborhood N_p of a and a positive real k_p such that $p(f(x))\leq k_p$ for $x\in N_p$; (ii) f is quasi-locally bounded at a with respect to S if, for each $B\in S$, there is a bounded $C_B\subset Y$ and a positive real ρ_B such that $f(a+th)\in C_B$ for $|t|<\rho_B$, $h\in B$. Then Hille's well-known theorem on the Fréchet-differentiability of Gâteaux-analytic functions extends in two directions, one involving Fréchet-Hyers differentiability and local quasi-boundedness, the other (for suitably restricted S) differentiability with respect to S and quasi-local boundedness with respect to S.

As a preliminary, the author develops in detail an extension of Campos Ferreira's theory of infinitesimal orders [Portugal. Math. **14** (1955), 43–62; MR **17**, 831].

J. H. Williamson (Cambridge, England)

284:

Wolfson, Kenneth G. Two-sided ideals of the affine near-ring. Amer. Math. Monthly **65** (1958), 29–30.

Let N be the set of all affine transformations of a vector space A over a division ring F. As noted by Blackett [Proc. Amer. Math. Soc. **7** (1956), 517–519; MR **17**, 1225], N is a near-ring with commutative addition. Let S be the set of constant transformations. It is known [see the cited review] that S is a maximal two-sided near-ring ideal of N if and only if A is finite-dimensional over F.

Let T denote the ring of all linear transformations of A over F. For each ordinal $\nu\geqq 0$ let T_ν be the subset of those transformations whose range has dimension less than $\aleph_\nu$ and set $T_{-1}=0$. It is shown that (a) for each $\nu\geqq -1$, $T_\nu+S$ is a two-sided near-ring ideal of N and this is the general form for such ideals; (b) $N/(T_\nu+S)$ is isomorphic to T/T_ν. From (b) it follows that every homomorphic (non-isomorphic) image of N is actually a ring.

B. Yood (Eugene, Ore.)

285:

Černy, Il'ya [Černý, Ilja]. On the extension of a linear operator in a topological linear space. Czechoslovak Math. J. **8** (**83**) (1958), 167–189. (Russian. English summary)

Let P be a locally convex topological linear space and let A be a linear operator defined on a linear subspace Q of P, as a limit of some directed set $\{A_h\}$ of continuous linear operators on P.

The author proves that P can be continuously imbedded into some larger convex topological space P^* in which the extension A^* of A is continuous, provided there are sufficiently many linear functionals on P of a certain kind. The topology of P^* restricted to P is weaker than the original topology of P.

Then the author gives sufficient conditions that the space P^* be minimal. It turns out that these conditions are satisfied if P is the set of all continuous real valued functions of one real variable and A is the differential operator. Thus there exists a one-to-one continuous linear correspondence between P^* and the space D_0 of all Schwartz' distributions.

P. Saworotnow (Washington, D.C.)

286:

★**Kaplansky, Irving. Functional analysis.** Some aspects of analysis and probability, pp. 1–34. Surveys in Applied Mathematics. Vol. 4. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xi+243 pp. $9.00.

This is a highly interesting short survey on some problems and results in functional analysis. Ch. 1: Topological linear spaces. It begins by reviewing the progress that has been made on some classical problems on Banach space. Then the principal classes of locally complex linear spaces (F, LF, barreled, bornological, Montel, nuclear) and the concept of Schwartz's distributions are defined and discussed. Ch. 2: Operators on topological linear spaces. First the classification problems (equivalence, similarity, congruence, unitary equivalence) are treated; then some special results are mentioned, mainly for Hilbert space operators (theorems of Fuglede, Heinz, Naimark, and others). Ch. 3: Banach algebras. It discusses several problems and results on commutative Banach algebras, C^*-algebras, W^*-algebras, and derivations; pertinent work of Šilov is reviewed here with particular emphasis. Some of the shorter proofs are reproduced or sketched, and some results are complemented by original remarks. Ch. 4: Group representations. This chapter is subdivided into 3 sections: General problems; semidirect products and induced representations; semisimple Lie groups. In particular, the work of Gelfand and Naimark,

Mackey, Harish-Chandra, and the author is discussed. Some open questions (partly with conjectured solutions) are formulated. The bibliography contains 113 items.

B. Sz.-Nagy (Szeged)

287:

Schäffer, Juan Jorge. Function spaces with translations. Math. Ann. **137** (1959), 209–262.

This paper contains a very detailed study of certain classes of function spaces; the work is related to that of Lorentz and Wertheim [Canad. J. Math. **5** (1953), 568–575; MR **15**, 324], Ellis and Halperin [ibid. **5** (1953), 576–592; MR **15**, 439], and Luxemburg and Zaanen [Indag. Math. **18** (1956), 110–119, 217–228; MR **17**, 987, 1113]. There are so many definitions, theorems, examples and counterexamples that it is impossible to state them all in this review; only some of the main results will be mentioned.

If Y is a normed linear space, contained in a locally convex Hausdorff topological vector space L, then Y is called stronger than L if the norm topology of Y is stronger than the topology induced in Y by L [equivalently: the unit sphere $\Sigma(Y)$ of Y is L-bounded]. In the particular case that L is itself a normed space, say $L=Z$, then $Y\leq Z$ means that Y is stronger than Z and $\Sigma(Y)\subset\Sigma(Z)$. Let $\mathscr{N}(L)$ be the class of all normed spaces stronger than the fixed space L, and let $\Gamma(L)$ be the class of all radially closed, L-bounded, balanced convex sets in L. There is a one-one correspondence between these classes, in which to each $Y\in\mathscr{N}(L)$ corresponds its unit sphere. Conversely, every set $K\in\Gamma(L)$ is the unit sphere of some such Y. Furthermore, $Y\leq Z$ if and only if the unit spheres satisfy $\Sigma(Y)\subset\Sigma(Z)$. The class $\Gamma(L)$ is a lattice under set inclusion, with $K_1\wedge K_2=K_1\cap K_2$, and $K_1\vee K_2$ the radial closure of the convex hull of $K_1\cup K_2$, so it follows (by the correspondence) that $\mathscr{N}(L)$ is a lattice with respect to $\leq$. The Banach spaces in $\mathscr{N}(L)$ form a sublattice.

The L-closure of a set in $\Gamma(L)$ lies again in $\Gamma(L)$; hence, applying this to the unit sphere of $Y\in\mathscr{N}(L)$, one obtains the unit sphere of another normed space in $\mathscr{N}(L)$, called the local closure of Y, and denoted by $\mathrm{cl}\,Y$. The space Y is called locally closed whenever $Y=\mathrm{cl}\,Y$, and quasi locally closed whenever Y is norm-equivalent to a locally closed space. Theorem: If L is complete and Y is quasi locally closed, then Y is a Banach space.

The foregoing results are applied in the case that L is the set of all Lebesgue measurable functions on $J=[0,\infty)$ with the topology of convergence in the mean on every bounded interval $J'\subset J$. Then L is locally convex, Hausdorff and complete. The class $\mathscr{F}$ consists of all normed spaces $F\in\mathscr{N}(L)$ satisfying in addition: If $f\in F$, g is measurable and $|g(t)|\leq|f(t)|$ on J, then $g\in F$ and $\|g\|\leq\|f\|$ (normality). The class $\mathscr{F}$ is a sublattice of $\mathscr{N}(L)$, and the Banach spaces in $\mathscr{F}$ form a sublattice of $\mathscr{F}$. The space $F\in\mathscr{F}$ is quasi locally closed if and only if, for every F-bounded increasing sequence $\{f_n\}$ of non-negative functions in F, its L-limit f lies in F. The space is locally closed if and only if, in addition, $\|f\|=\lim\|f_n\|$ holds for the F-norms. Finite joins and meets of (quasi) locally closed spaces in $\mathscr{F}$ are (quasi) locally closed.

For any measurable $f(t)$ on $J=[0,\infty)$ and any $\tau\geq0$, the translation operators $T_\tau{}^+$ and $T_\tau{}^-$ are defined by $T_\tau{}^+f(t)=f(t-\tau)$ for $t\geq\tau$ and 0 for $0\leq t<\tau$; $T_\tau{}^-f(t)=f(t+\tau)$ for $t\geq0$. The subclass $\mathscr{T}^\#$ of $\mathscr{F}$ consists of those $F\in\mathscr{F}$ for which $F\neq\{0\}$, and $f\in F$ implies $T_\tau{}^+f$, $T_\tau{}^-f\in F$ with $\|T_\tau{}^+f\|=\|f\|$. All Orlicz spaces (on J) lie in $\mathscr{T}^\#$, and they are locally closed. However, not every locally closed space in $\mathscr{T}^\#$ is (norm-equivalent to) an Orlicz space. $\mathscr{T}^\#$ is a sublattice of $\mathscr{F}$, and the class of locally closed spaces in $\mathscr{T}^\#$ is a sublattice of $\mathscr{T}^\#$. The spaces $F\in\mathscr{T}^\#$ have no unfriendly sets (the set $E\subset J$ of positive measure is called unfriendly if $f=0$ a.e. on E for all $f\in F$). For any $F\in\mathscr{T}^\#$, the associate space F' is defined in the usual manner: $g(t)\in F'$ whenever $\int_0^\infty|f(t)g(t)|dt\leq k$ (k constant) for all $\|f\|\leq1$. The F'-norm is the supremum of the integral on the left. Then $F'\in\mathscr{T}^\#$, $F'=(\mathrm{lc}\,F)'$, and F' is locally closed. Furthermore, $F''=\mathrm{lc}\,F$; hence $F''=F$ if and only if F is locally closed.

There are further applications to Orlicz spaces and spaces of continuous functions; the results are extended to normed spaces $F(X)$ of functions on J with values in the Banach space X.

A. C. Zaanen (Leiden)

288:

Régnier, André. Variétés maximales, homogénéité ergodique. C. R. Acad. Sci. Paris **248** (1959), 914–916.

The author considers the notion of maximal linear variety in a Banach space with particular reference to certain closed sets associated with a bounded family of elements of the space. These considerations are then applied to abelian semi-groups of contractions of a Banach space. Such notions as ergodically null element and ergodically homogeneous semi-group are discussed. {Note: It seems to the reviewer that, in definition 1, V should be required to be closed and a proper subset of the Banach space. A similar remark applies to lemma 1.}

A. F. Ruston (Sheffield)

289:

Dixmier, J. Sur la relation $i(PQ-QP)=1$. Compositio Math. **13** (1958), 263–269.

On the standard infinite sequence Hilbert space, consider the symmetric operators P', Q' represented by the matrices

$$(P')_{jk} = -i2^{-\frac12}(j^{\frac12}\delta_{j+1,k}-k^{\frac12}\delta_{j,k+1}),$$
$$(Q')_{jk} = 2^{-\frac12}(j^{\frac12}\delta_{j+1,k}+k^{\frac12}\delta_{j,k+1})$$

and satisfying the commutation relation $P'Q'-Q'P'=-iI$. It is known that the corresponding least closed extensions, P'' and Q'', the Heisenberg operators, are self-adjoint; that is, P' and Q' are essentially self-adjoint. Moreover, the system $\{P'',Q''\}$ is irreducible. The question of the uniqueness, to within unitary equivalence, of an irreducible pair of closed, symmetric operators P,Q satisfying $PQ-QP=-iI$ on a dense set X of a Hilbert space is considered. In the case of one degree of freedom, the author's result (theorem 1) generalizes that of Rellich [Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 107–115; MR **9**, 192] and is, essentially, that P and Q are unitarily equivalent to the Heisenberg operators in case the contraction of P^2+Q^2 to X is essentially self-adjoint. A similar result (theorem 2) is obtained in the case of several degrees of freedom, but it is pointed out that this theorem does not contain Rellich's result in this instance.

C. R. Putnam (Lafayette, Ind.)

290:

Domračeva, G. I. Ideals in normal subrings of the ring of continuous functions. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 29–38. (Russian)

The set of all continuous functions from a compact space Q into the reals, with $\pm\infty$ adjoined, that assume finite values except on a nowhere dense subset of Q is denoted by $C_\infty(Q)$. It is assumed throughout the paper that the closure of every open F_σ in Q is open, so that pointwise addition and multiplication of functions can be defined in the natural way, making $C_\infty(Q)$ into a ring. A subring $C_\infty^0(Q)$ of $C_\infty(Q)$ is called normal if $x\in C_\infty^0(Q)$ whenever $0\leqq x\leqq y$ and $y\in C_\infty^0(Q)$. For such subrings, the author proves analogues of the Gelfand-Kolmogoroff theorems [Dokl. Akad. Nauk SSSR **22** (1939), 11–15] which characterize the maximal ideals in rings of real-valued continuous functions on spaces that are not necessarily compact. *M. Jerison* (Lafayette, Ind.)

291:

Poulsen, Ebbe Thue. On the algebra generated by a continuous function. Math. Scand. **6** (1958), 37–39.

In answer to a question posed to the author by W. G. Bade, it is shown that there does belong to the algebra A of continuous real-valued functions on the closed unit interval a function g such that g fails to distinguish points of the interval and yet g has the property that if $f\in A$ and

$$\int_0^1 f(x)g^n(x)dx = 0 \qquad (n=0,1,2,\cdots),$$

then f vanishes identically.

T. A. Botts (Charlottesville, Va.)

292:

Szeptycki, P. A simple proof of an imbedding theorem of the Kondrashev type. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 561–564.

In the present paper the author gives a new proof of a theorem of S. L. Sobolev [*Nekotorye primeneniya funkcional'nogo analiza v matematičeskoi fizike*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565; pp. 169–175] based on an inequality of L. Nirenberg [Comm. Pure Appl. Math. **9** (1956), 509–529; MR **19**, 962]. Given a bounded domain Ω in the Euclidean space E_N, let C denote the space of all continuous functions $f(x)$, $x\in\Omega$, and assume in C the uniform topology. Let C^∞ denote the space of all infinitely differentiable functions $f(x)$, $x\in\Omega$, and assume in C^∞ the topology defined by the norm $\|f\|_p^l$, where $\|f\|_p^l=\sum(\Omega)\int|D^sf(x)|^pdx$, $\sum$ ranges over all partial derivatives $D^sf(x)$ of $f(x)$ of order $s=(s_1,\cdots,s_N)$, $0\leqq s_1+\cdots+s_N\leqq l$, and $l\geqq 0$, $p\geqq 0$ are given. The domain Ω is supposed to satisfy a condition considered by L. Nirenberg [ibid. **8** (1955), 649–675; MR **17**, 742] and different from the original one. Sobolev's theorem states that if $l\geqq[N/p]+1$, $p>1$, then the completion W_p^l of C^∞ in the sense of the topology $\|f\|_p^l$ is contained in C, and the imbedding $W_p^l\to C$ is completely continuous in the sense of the chosen topologies. *L. Cesari* (Baltimore, Md.)

293:

Myškis, A. D.; and Lepin, A. Ya. On the definition of generalized functions. Mat. Sb. N.S. **43** (**85**) (1957), 323–348. (Russian)

This paper continues the approach to Schwartz distributions of Korevaar [Indag. Math. **17** (1955), 663–674; MR **17**, 594], Mikusiński [Bull. Acad. Polon. Sci. Cl. III **3** (1955), 589–591; MR **17**, 594], and Temple [Proc. Roy. Soc. London Ser. A **228** (1955), 175–190; MR **16**, 910]. Let $\mathfrak{M}$ be any class of non-negative functions $m(x)$, $-\infty<x<\infty$, such that: (1) $\mathfrak{M}$ is closed under addition; (2) $1\in\mathfrak{M}$; (3) $m(x)\in\mathfrak{M}$ implies $|x|m(x)\in\mathfrak{M}$, $\max_{0\leq t\leq 1} m(tx)\in\mathfrak{M}$, and also $m_1(x)\in\mathfrak{M}$ whenever $0\leqq m_1(x)\leqq m(x)$. {It seems to the reviewer that another condition, such as continuity of the functions $m(x)$, is implicitly assumed.} A sequence $F_n(x)$ is said to converge $\mathfrak{M}$-almost uniformly to $F(x)$ if it converges uniformly to $F(x)$ on every finite interval and $\sup_n|F_n(x)|\in\mathfrak{M}$. A sequence of functions $f_n(x)$ is $\mathfrak{M}$-fundamental if there is a positive integer k and an $\mathfrak{M}$-almost uniformly convergent sequence of k-times continuously differentiable functions $F_n(x)$ such that $F_n^{(k)}=f_n$. $\mathfrak{M}$-generalized functions are equivalence classes of $\mathfrak{M}$-fundamental sequences. If $\mathfrak{M}$ is the smallest class $\mathfrak{M}^0$ satisfying (1)–(3), the space $M_{\mathfrak{M}}$ of $\mathfrak{M}$-generalized functions is in effect Schwartz's space of slowly increasing distributions [*Théorie des distributions*, vol. II, Hermann, Paris, 1951; MR **12**, 833; p. 95], while if $\mathfrak{M}$ is the largest such class $\mathfrak{M}^1$, $M_{\mathfrak{M}}$ consists of all distributions of finite order. Let $M_{\mathfrak{M}}$ have the pseudotopology of $\mathfrak{M}$-almost uniform convergence for sequences of primitives $F_n(x)$. Let $O_{\mathfrak{M}}$ be the space of all infinitely differentiable functions $s(x)$ such that

$$s^{(k)}(x)m(x)\xrightarrow[x\to\infty]{}0,\quad k=0,1,2,\cdots,\quad \text{all } m(x)\in\mathfrak{M}.$$

Write $s_n(x)\to s(x)$ ($\mathfrak{M}$) if, for each $k=0,1,2,\cdots$, $s_n^{(k)}(x)$ tends uniformly to $s^{(k)}(x)$ and

$$m(x)\sup_n|s_n^{(k)}(x)|\xrightarrow[x\to\infty]{}0,\quad \text{all } m(x)\in\mathfrak{M}.$$

Theorem: $O_{\mathfrak{M}}$ is the dual of $M_{\mathfrak{M}}$. Moreover, $s_n(x)\to s(x)$ ($\mathfrak{M}$) if and only if s_n tends weakly to s. Theorem: $M_{\mathfrak{M}}$ is the dual of $O_{\mathfrak{M}}$ provided there exists a countable sequence $m_i(x)\in\mathfrak{M}$ such that, for every $m(x)\in\mathfrak{M}$, $m(x)\leqq m_i(x)$ for some i. This countability condition holds, for example, if $\mathfrak{M}=\mathfrak{M}^0$ but not if $\mathfrak{M}=\mathfrak{M}^1$.

W. H. Fleming (Providence, R.I.)

294:

Mişkis, A. D. [Myškis, A. D.]; and Lepin, A. I. Definition of distributions. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 3 (26), 15–39. (Romanian)

Romanian translation of the article reviewed above.

295:

König, Heinz. Eine Charakterisierung der Distributionen endlicher Ordnung. Math. Ann. **136** (1958), 240–244.

Let D be the linear space of infinitely often differentiable complex-valued functions on the real Euclidean n-space R^n having compact support. The m-norm $\|\phi\|_m$ of an element ϕ of D is defined as sup $|D^p\phi(x)|$ as D^p varies over all derivatives of order $\leqq m$, and x over R^n. If D_ρ denotes the subset of those $\phi\in D$ whose support is contained in the closed ball of radius ρ with the origin as center, then a distribution S (in the sense of Schwartz) is called of finite order m if for each positive ρ the sup of $|S(\phi)|$, as ϕ varies over those elements of D_ρ whose m-norm is not greater than 1, is finite. This sup is denoted by $\|S\|_m(\rho)$.

The author shows first that the following fact is nearly obvious: to each distribution S of finite order m and each $\rho>0$ there exists a non-negative monotone increasing function H on $[0,\infty)$ such that for all $\phi\in D_\rho$

$$(1)\qquad (S*\phi)(x)=O(H(|x|))\quad (|x|\to\infty).$$

The main result of the paper is the following converse: if S is a distribution on R^n and if to S and to each $\rho > 0$ there exists a function H of the properties described above for which (1) holds for all $\phi \in D_\rho$, then S is of finite order for some m. Moreover, in case $H \neq 0$ the asymptotic relation

$$\|S\|_m(r) = O((r+\rho)^n H(r+\rho)) \quad (r \to \infty)$$

holds for large enough m.

E. H. Rothe (Ann Arbor, Mich.)

296:

Sebastião e Silva, J. Sur l'espace des fonctions holomorphes à croissance lente à droite. Portugal. Math. **17** (1958), 1–17; errata **18** (1959), 154–155.

This paper is a sequel to, and to some extent a commentary on, a previous paper by the same author [Portugal. Math. **14** (1956), 105–132; MR **18**, 137]. The space $\mathfrak{A}_\infty$ is the strict inductive limit of the spaces $\mathfrak{A}_k$ ($k = 0, 1, 2, \cdots$), where $\mathfrak{A}_k$ is the Banach space of all functions ϕ such that $\phi(z)/z^k$ is holomorphic when Real $(z) > k$, continuous and bounded when Real $(z) \geq k$, and the norm $\|\phi\|_k$ is the supremum of $|\phi(z)/z^k|$ for Real $(z) \geq k$. For a given λ the function of z given by $(z-\lambda)^{-1}$ is in $\mathfrak{A}_\infty$. It is denoted by $\mathbf{h}(\lambda)$. Let E be a sequentially complete locally convex topological linear space, and let F be a continuous linear mapping of $\mathfrak{A}_\infty$ into E. Then F is representable by an integral involving the "indicatrix" $\mathbf{f}(\lambda) = F[\mathbf{h}(\lambda)]$. In the present paper the author makes use of a suggestion of Laurent Schwartz to show that a function $\mathbf{f}$ from the complex plane to E is such an indicatrix if and only if (1) it is an entire function of λ, and (2) for each non-negative integer k there exist elements $\mathbf{a}_1, \cdots, \mathbf{a}_k$ in E and a bounded set M_k in E such that

$$\mathbf{f}(\lambda) - \frac{\mathbf{a}_1}{\lambda} - \frac{\mathbf{a}_2}{\lambda^2} - \cdots - \frac{\mathbf{a}_k}{\lambda^k} \in \frac{1}{\lambda^{k+1}} M_k$$

when Real $(\lambda) \leq k$.

Next, suppose further that the space E is an algebra, with unit $\mathbf{e}$, in which the product of two elements is continuous in each element separately. Suppose further that F is not only continuous linear, but is a homomorphism (i.e. product-preserving), and suppose F maps the constant function 1 into $\mathbf{e}$. Let F map the function z into $\mathbf{a}$. Then the indicatrix in this case is $(\mathbf{a} - \lambda\mathbf{e})^{-1}$, and by setting $F(\phi) = \phi(\mathbf{a})$, an operational calculus results. The elements $\mathbf{a}$ of the algebra arising in this way from homomorphisms F are characterized by three conditions: (1) $\mathbf{a} - \lambda\mathbf{e}$ has an inverse in the algebra, for every λ; (2) the inverse $(\mathbf{a} - \lambda\mathbf{e})^{-1}$ is locally bounded as a function of λ; and (3) $(\mathbf{a} - \lambda\mathbf{e})^{-1}$ is bounded on every left half-plane Real $(\lambda) \leq k$.

Applications to the theory of distributions and to operational calculus for differential operators are considered.

A. E. Taylor (Los Angeles, Calif.)

297:

Kadec, M. I. Linear dimension of the spaces L_p and l_q. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 95–98. (Russian)

The question under consideration is whether there is an isomorphic embedding of l_q in L_p. With the posthumous publication of a paper of Paley [Bull. Amer. Math. Soc. **42** (1936), 235–240], this question was settled for all cases excepting $1 \leq p < q < 2$. In most cases, no embedding is possible, and apparently Paley thought that it was not possible in case $1 \leq p < q < 2$ either. In the present note, however, an explicit embedding of l_q into L_p is constructed for this case. The construction is based upon the representation of L_p with respect to a probability distribution function $F(x)$ determined by the condition

$$\int_{-\infty}^{\infty} \cos(tx) dF(x) = e^{-|t|^q}.$$

M. Jerison (Lafayette, Ind.)

298:

Musielak, J.; and Orlicz, W. On modular spaces. Studia Math. **18** (1959), 49–65.

Easing the hypothesis of the "modular" which was defined first by the reviewer [J. Fac. Sci. Univ. Tokyo, Sect. I. **6** (1951), 85–131; MR **13**, 362], the authors attempt a generalization of modulared linear spaces such that the new modular is related to Fréchet spaces, while the original gives a Banach space. A functional $\rho(x)$ defined on a linear space with values $-\infty < \rho(x) \leq \infty$ will be called a modular if the following conditions hold: (1) $\rho(x) = 0$ if and only if $x = 0$; (2) $\rho(-x) = \rho(x)$; (3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every α, $\beta \geq 0$, $\alpha + \beta = 1$; (4) $\rho(\alpha x) < \infty$ for some $\alpha > 0$; (5) $\lim_{\nu\to\infty} \alpha_\nu = 0$ implies $\lim_{\nu\to\infty} \rho(\alpha_\nu x) = 0$. For such a modular $\rho(x)$, putting

$$\|x\| = \inf\{\varepsilon > 0: \ \rho(x/\varepsilon) < \varepsilon\}$$

we obtain a F-norm, i.e.: (1) $\|x\| = 0$ if and only if $x = 0$; (2) $\|x+y\| \leq \|x\| + \|y\|$; (3) $|\alpha| \leq |\beta|$ implies $\|\alpha x\| \leq \|\beta x\|$; (4) $\lim_{\nu\to\infty} \alpha_\nu = 0$ implies $\lim_{\nu\to\infty} \|\alpha_\nu x\| = 0$. The authors investigate by examples when a modular fulfills the condition: $\lim_{\nu\to\infty} \rho(x_\nu) = 0$ implies $\lim_{\nu\to\infty} \rho(\alpha x_\nu) = 0$ for every real α.

H. Nakano (Sapporo)

299:

Suharevskiĭ, I. V. On the λ-stability of solutions of operator equations in Banach space. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 454–457. (Russian)

Let a linear completely continuous operator $A_{(\lambda)}$ be defined throughout a simply connected domain of the complex plane, depending analytically on λ in the sense that for each λ_0 in the domain the series $A_{(\lambda)} = \Sigma(\lambda - \lambda_0)^j A_j$ is convergent in norm in some neighbourhood of λ_0, and the A_j are bounded operators. If $A_{(\lambda)}$ has at least one regular point, then $R_{(\lambda)} = (I - A_{(\lambda)})^{-1}$ is meromorphic. Let F_λ be the set of f for which the equation (1): $u - A_{(\lambda)}u = f$, is soluble; the solution will be written u_λ at a regular point. If, for each $f \in F_\lambda$, u_λ tends to a limit $u_{(\lambda_0)}$ as $\lambda \to \lambda_0$, then λ_0 is called a point of stability of $A_{(\lambda)}$. $u_{(\lambda_0)}$ is a solution of the equation (1) for $\lambda = \lambda_0$ which is called λ-stable. It is shown that every regular point and every simple pole of R is a point of stability. At a simple pole the λ-stable solution of (1) for $\lambda = \lambda_0$ is selected uniquely from the others by the condition that $A_1 u_{(\lambda_0)}$ is orthogonal to $P^*_{(\lambda_0)}$, the proper space of $A^*_{(\lambda_0)}$. If λ_0 is a multiple pole and the functionals $V_j(A_1 f)$, where $\{V_j\}$ is a basis for $P^*_{(\lambda_0)}$, are independent, then λ_0 is not a point of stability. For a pole λ_0 of R to be simple it is necessary and sufficient that if $\{u_i\}$ is a basis for the proper space of $A_{(\lambda_0)}$, then $\det \|V_j(A_1 u_i)\| \neq 0$. If λ_0 is a proper value of $A_{(\lambda)}$ of rank 1, then a necessary and sufficient condition that λ_0 be a point of stability is that, if s is the smallest number such that $V_1(A_s f)$ is not identically 0, then $V_1(A_s u_1) \neq 0$. The λ-stable solution of (1) at λ_0 is characterized by the property that $V_1(A_s u_{(\lambda_0)}) = 0$.

J. L. B. Cooper (Cardiff)

300:

Schröder, Johann. Störungsrechnung bei Eigenwert- und Verzweigungsaufgaben. Arch. Rational Mech. Anal. **1** (1958), 436–468.

The author considers the problem in a Banach space of determining the constant λ and the element ϕ which satisfy the equation $A_0\phi - \lambda\phi = B(\lambda, \phi)$, subject to $l\phi = 1$, where A_0 is a linear operator, l a bounded linear functional. He considers the case that the unperturbed problem (found by setting $B \equiv 0$) has the solution $\lambda = \mu_0$, $\phi = \psi_0$ (a simple eigenvalue and its eigenvector). His perturbation method uses an iteration procedure, assumes that B is "small", and finds a solution in the neighborhood of the unperturbed solution. If B is linear in ϕ, the problem is an eigenvalue problem; if B is non-linear it is called a bifurcation problem. The author obtains quite good estimates of the error at any stage of the iteration procedure. He first gives the abstract derivation of his results and then an analytical derivation for the case of matrix problems and for the case of second order ordinary differential equation problems. He finally gives numerical examples for the matrix case and for the buckling of a rod (a bifurcation problem), which illustrate the precision of his error estimates.

The method of analyzing the iteration procedure is based on earlier work of the author [Arch. Rational Mech. Anal. **1** (1957), 154–180; MR **20** #2072].

E. Isaacson (New York, N.Y.)

301a:

Gel'man, I. V. Some functional spaces and their application to variational problems. Dokl. Akad. Nauk SSSR **122** (1958), 547–550. (Russian)

301b:

Gel'man, I. V. On a certain non-linear operator. Dokl. Akad. Nauk SSSR **120** (1958), 454–456. (Russian)

In these papers the author studies a non-linear integro-differential operator in the setting of Orlicz spaces.

M. M. Day (Urbana, Ill.)

302:

Akilov, G. P.; and Veršik, A. M. On the reciprocal continuous extension of linear operations. Vestnik Leningrad. Univ. **13** (1958), no. 7, 27–33. (Russian. English summary)

Let X_0 be a Banach space and U_0 a one-to-one linear operator carrying X_0 into itself. A locally convex space X, the inductive limit of a sequence of copies of X_0 under the sequence of copies of U_0 interpreted as carrying one copy of X_0 into the next, and its related mapping U are shown to be such that U is one-to-one from X onto X and has a continuous inverse. When U_0 is, instead of one-to-one, assumed to have a dense range, the above calculation can be carried out for the dual operator U_0^*. The results are applied to distributions [Schwartz, *Théorie des distributions*, vol. I, Hermann, Paris, 1950; MR **12**, 31] and to a space of set-functions.

M. M. Day (Urbana, Ill.)

303:

Vaĭnberg, M. M.; and Èngel'son, Ya. L. The square root of a linear operator in locally convex spaces. Dokl. Akad. Nauk SSSR **122** (1958), 755–758. (Russian)

Vaĭnberg [Dokl. Akad. Nauk SSSR **100** (1955), 845–848; MR **16**, 934] and Èngel'son [Uč. Zap. Latv. Gos. Univ. **8** (1956), 73] have investigated square roots of linear operators by assuming that the operators are completely continuous. The main generalization accomplished by this article consists in dispensing with the complete continuity condition. Let E be a real locally convex vector space whose dual E' is subjected to the strong topology. Let $S < F$ indicate that $S \subset F$ and that the topology of S majorizes the topology induced in S by F. As usual, $\langle x, y\rangle = y(x)$ when $y \in E'$ and $x \in E$. The basic assumption is the existence of a Hilbert space H such that $E < H < E'$ and whose scalar product coincides with $\langle x, y\rangle$ when $y \in H$ and $x \in E \cup E''$; consequently, the bidual E'' must be a subset of H. This situation is exemplified by $E = L^p(B)$ and $H = L^2(B)$ when $p > 2$ and B is a set of finite measure. Let $\mathscr{L}(S, F)$ be the set of continuous linear mappings of S into F. Suppose that $A \in \mathscr{L}(E', E)$; then A can be identified with a member A_1 of $\mathscr{L}(H, H)$, which is required to be positive and self-adjoint.

Basic result: the positive square root B of A_1 belongs to $\mathscr{L}(H, E'')$ and admits a continuous extension $\tilde{B}$ in $\mathscr{L}(E', H)$. Next, E is a quasi-t-space (t=tonnelé) by assumption. Then $A = B\tilde{B}$. Moreover, A admits a continuous extension $\bar{A}$ in $\mathscr{L}(E''', E'')$ and $\bar{A} = BB'$, where $B' \in \mathscr{L}(E''', H)$. This extends a Banach space result of V. I. Sobolev [Dokl. Akad. Nauk SSSR **111** (1956), 951–954; MR **18**, 912]. The article includes several variants of the above propositions.

G. L. Krabbe (Lafayette, Ind.)

304:

Stefanova, L. B. An ordinary differential operator depending on a parameter. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 231–235. (Russian)

The differential operator

$$A_n^{(\alpha)}u = (-1)^n d^n/dt^n (t^\alpha d^n u/dt^n)$$

is considered on the space of functions $u \in C^{2n}[0, 1]$ which satisfy certain boundary conditions selected, according to the value of the non-negative parameter α, as follows: In any case $u(1) = u'(1) = \cdots = u^{(n-1)}(1) = 0$; if $\alpha < n$ then the conditions $u(0) = \cdots = u^{(n-k-1)}(0) = 0$ are also imposed, where $k = [\alpha]$. With this choice of boundary conditions $A_n^{(\alpha)}$ is symmetric and non-negative:

$$(A_n^{(\alpha)}u, u) = \int_0^1 t^\alpha (d^n u/dt^n)^2 dt.$$

Use the same symbol $A_n^{(\alpha)}$ to denote the self-adjoint extension (in the sense of Friedrichs) of $A_n^{(\alpha)}$ to $L_2(0, 1)$. Then: (1) If $\alpha < 2n$ then $A_n^{(\alpha)}$ has pure point spectrum and lower bound $\geq m_n = 1^2 3^2 \cdots (2n-1)^2/4^2$; (2) if $\alpha > 2n$ then $A_n^{(\alpha)}$ does not have pure point spectrum and its lower bound is 0; (3) in the case $\alpha = 2n$ the spectrum of $A_n^{(\alpha)}$ is purely continuous and consists of the ray $[m_n, \infty)$. This work forms a continuation of that of Mihlin [Vestnik Leningrad. Gos. Univ. **9** (1954), no. 8, 19–48; MR **17**, 493] in which these results are obtained (apparently with somewhat greater precision) for the case $n = 1$.

A. Brown (Houston, Tex.)

305:

Nussbaum, A. E. Integral representation of semi-groups of unbounded self-adjoint operators. Ann. of Math. (2) **69** (1959), 133–141.

A semi-group $\mathfrak{S}$ is called a locally compact full semi-group if (a) $\mathfrak{S}$ can be embedded in a locally compact

group $\mathfrak{G}$, (b) $\mathfrak{S}$ is locally compact in the relative topology of $\mathfrak{G}$, and (c) every non-empty bounded open set in $\mathfrak{S}$ has non-zero measure with respect to the left (or right) Haar measure of $\mathfrak{G}$. Given such a semi-group $\mathfrak{S}$, suppose that to each $x \in \mathfrak{S}$ there corresponds a self-adjoint operator T_x acting in a fixed Hilbert space $\mathfrak{H}$, such that (i) $T_{xy} \subseteqq T_x T_y$, (ii) for any $u \in \mathfrak{D} = \bigcap_{x \in \mathfrak{S}} \mathfrak{D}(T_x)$ and $v \in \mathfrak{H}$, $(T_x u, v)$ is a continuous function of x on $\mathfrak{S}$ [$\mathfrak{D}(T_x)$ denotes the domain of definition of T_x in $\mathfrak{H}$]. By a joint paper of A. Devinatz, the author, and J. von Neumann [same Ann. **62** (1955), 199–203; MR **17**, 178], we have then $T_{xy} = T_x T_y$. Suppose also that there exists a denumerable set $D \subset \mathfrak{S}$ such that, for each $x \in \mathfrak{S}$, $x\mathfrak{S} \cap D$ or $\mathfrak{S}x \cap D$ is non-void. Let $\hat{\mathfrak{S}}$ denote the semi-group of real characters χ of $\mathfrak{S}$ with the compact-open topology. The main result of the paper is theorem 6: There exists a spectral measure $\{E(\sigma)\}$ on the Borel subsets of $\hat{\mathfrak{S}}$ such that $T_x = \int_{\hat{\mathfrak{S}}} \chi(x) E(d\chi)$. If Q is a bounded operator which commutes with every T_x, $x \in \mathfrak{S}$, then Q commutes with every $E(\sigma)$. This theorem is a generalization of previous results of B. Sz.-Nagy, E. Hille, A. Devinatz, and the author, concerning in particular weakly continuous one-parameter semi-groups of bounded self-adjoint operators.

B. Sz.-Nagy, W. Zawadowski (Szeged, Warsaw)

306:

Ionescu Tulcea, Cassius. Spectral representation of certain semi-groups of operators. J. Math. Mech. 8 (1959), 95–109.

Let G be a locally compact group, S a locally compact semi-group in G for which every non-void open set in S has positive Haar measure. A character on S is a continuous complex-valued function $\chi(\not\equiv 0)$ which satisfies $\chi(s)\chi(t) = \chi(st)$ for every $s, t \in S$. Let E be the collection of characters endowed with the topology of uniform convergence on compact sets. For any locally bounded function $r(s) \geqq 0$, defined on S, let $E(r)$ be the subset of E for which $|\chi(s)| \leqq r(s)$. The author shows that $E(r)$ is locally compact and also gives sufficient conditions under which E is locally compact.

The principal result of the paper concerns the integral representation of semi-groups of normal operators (not necessarily bounded) which act on a Hilbert space X. Let $\{U_s; s \in S\}$ be a set of normal operators on X which satisfy the conditions: (1) $U_{st} \subset U_s U_t$; (2) U_s and U_t commute; (3) $(U_s x|y)$ is continuous for every $x \in \bigcap D(U_s)$ and $y \in X$. (In the case where the U_s are not bounded operators, more conditions are required on S.) Then there exists a unique Hermitian spectral family $\{\mu_{x,y}; x, y \in X\}$ on E such that $(U_s x|y) = \int_E \chi(s) d\mu_{x,y}(\chi)$ In case the operators are all bounded, integration over E may be replaced by integration over an $E(r)$.

This theorem was previously announced by the author in a weaker form [Proc. Nat. Acad. Sci. **44** (1958), 44–45; MR **20** #2628]. At about the same time, and using essentially the same methods, A. E. Nussbaum [Amer. Math. Soc. Notices **5** (1958), 95; and the paper reviewed above] had obtained the sharper version given in this paper. The theorem contains as special cases representation theorems of Stone, Hille, B. Sz.-Nagy, Phillips, Devinatz, Getoor, Nussbaum, and Devinatz and Nussbaum. References for the latter papers will be found in the present paper. *A. Devinatz* (St. Louis, Mo.)

307:

Arsen'ev, G. I. Algebras of linear operators in Hilbert space. Uč. Zap. Borisoglebsk. Gos. Ped. Inst. **1958**, no. 5, 119–132. (Russian)

Expository.

308:

Getoor, R. K. An anologue of Mercer's theorem. Duke Math. J. **25** (1958), 615–624.

An expansion theorem for continuous, Hermitian symmetric, positive definite Carleman kernels is proved, which is analogous to Mercer's theorem for Hilbert-Schmidt kernels. The expansions used are the generalized eigenfunction expansions for self-adjoint operators studied by Mautner [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 49–53; MR **14**, 659]; L. Gårding [Institute for Fluid Dynamics and Applied Mathematics, Lecture Series no. 11, Univ. of Maryland, College Park, Md., 1954; MR **17**, 159]; W. Bade and J. Schwartz [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 519–525; MR **18**, 125]. *J. Elliott* (New York, N.Y.)

309:

Bahtin, I. A.; and Krasnosel'skiĭ, M. A. On the theory of equations with concave operators. Dokl. Akad. Nauk SSSR **123** (1958), 17–20. (Russian)

The notion of a concave operator has been defined by Bahtin [same Dokl. **117** (1957), 13–16; MR **20** #3465]. Essentially it is a non-linear operator defined on a Banach space and having properties that relate it to a partial ordering on the space. In the present paper there is announced the theorem that if a completely continuous concave operator A has a unique fixed point, then the sequence of successive approximations, $\phi_{n+1} = A\phi_n$, converges in norm to the fixed point. This result is almost independent of the initial point ϕ_0. Several refinements of this theorem are also announced, and an application is mentioned to a type of non-linear integral equation where positive solutions are desired.

G. Hufford (Seattle, Wash.)

GEOMETRY

See also 336, 337.

310:

Segre, Beniamino. Elementi di geometria non lineare sopra un corpo sghembo. Rend. Circ. Mat. Palermo (2) **7** (1958), 81–122.

Classical synthetic methods are applied to second degree problems in a (left or right) projective space over a noncommutative field. Define a quadric to be the set of points swept out by the lines (called the generators of the quadric) meeting each of three given mutually skew lines in 3-space, a conic to be the intersection of a quadric with a plane containing no generator. Noncommutativity of the ground field introduces a certain amount of pathology (the intersection of a quadric with a plane through a generator is complicated, and a line may meet a conic in more than two points), but some classical results survive. For example, any two quadrics (or conics) over the same field are homographic, and an analogue of Pascal's hexagon theorem holds. There is much working out of details and many enumerative results, depending on the cardinality of the field and its center. *M. Rosenlicht* (Evanston, Ill.)

311:

Szász, Paul. New proof of the circle axiom for two circles in the hyperbolic plane by means of the end-calculus of Hilbert. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 97–100.

J. C. H. Gerretsen [Nederl. Akad. Wetensch. Proc. **45** (1942), 360–366, 479–483, 559–566; MR **6**, 13; cf. p. 566] proved trigonometrically the circle axiom of the absolute geometry: "If a circle has a point outside and a point inside another circle in the plane, the circles meet in just two points." Here it is proved with help of the analytic geometry of the hyperbolic plane as established by the author [Acta Math. Acad. Sci. Hungar. **9** (1958), 1–28; MR **20** #4223] on the basis of Hilbert's "Endenrechnung" with direct introduction of the Weierstrass coordinates.

T. Takasu (Yokohama)

312:

Hofmann, Joseph E. Über sich nichttreffende hyperbolische Gerade. Arch. Math. **9** (1958), 219–227.

Let K be the unit circle and let $g_1 = U_1V_1$, $g_2 = U_2V_2$ (U_1, V_1, U_2, V_2 points on K) be two arcs of non-intersecting circles orthogonal to K (i.e. two non-intersecting lines in Klein's model of the hyperbolic geometry). Let u, v be the circles orthogonal to K through U_1, V_2, and U_2, V_1, respectively, and let S be their intersection point. The angle $\varphi = U_1SU_2$ is taken as the angle between g_1, g_2, and some elementary properties and constructions which generalizes those of the ordinary angle between intersecting lines are worked out.

L. A. Santaló (Buenos Aires)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 41, 111, 631, 632, 633, 635.

313:

Abrham, Jaromír. An approximative method for non-linear programming. Časopis Pěst. Mat. **83** (1958), 425–439. (Czech. Russian and English summaries)

The paper presents an iterative process for finding the minimum of a strictly convex function $f(X)$ on the set $\mathfrak{M}$ of non-negative solutions of a system of linear equations $\sum_{j=1}^{m+k} a_{ij}x_j = b_i$, $i = 1, \cdots, m$, $k \geqq 1$. To prove the convergence of the process the following criterion is given: a necessary and sufficient condition for $f(X_0) = \min_{x \in \mathfrak{M}} f(X)$ is that the point X_0 be minimal with respect to all vectors of some corresponding basis. The theory is then applied to the case when $f(X)$ is a sum of a linear form and a positive definite quadratic form. The paper concludes with a numerical example.

D. Mazkewitsch (Cincinnati, Ohio)

314:

Bílý, Josef; Fiedler, Miroslav; und Nožička, František. Die Graphentheorie in Anwendung auf das Transportproblem. Czechoslovak Math. J. **8** (**83**) (1958), 94–121. (Russian summary)

This paper describes a graph theoretical algorithm for the solution of the transportation problem of linear programming. It is the simplex method in disguise.

H. W. Kuhn (London)

315:

Goldstein, Allen A.; and Cheney, Ward. A finite algorithm for the solution of consistent linear equations and inequalities and for the Tchebycheff approximation of inconsistent linear equations. Pacific J. Math. **8** (1958), 415–427.

This paper offers exactly what its title announces, where the Tchebycheff approximation to the solution of the system of linear equations $Ax = b$ is an x which minimizes the maximum component of $|Ax - b|$. The algorithm is a method of descent which obtains the lowest vertex of a convex polytope.

H. W. Kuhn (London)

316:

Gaddum, Jerry W. Linear inequalities and quadratic forms. Pacific J. Math. **8** (1958), 411–414.

This note establishes several criteria for restricted notions of definiteness for quadratic forms in terms of the solvability of associated systems of linear inequalities. The following result is typical. A quadratic form $Z(x) = \sum a_{ij}x_ix_j$ is said to be conditionally (positive) definite if $Z(x) > 0$ whenever $x \geqq 0$. Theorem: Suppose each principal minor of $Z(x)$ is conditionally definite. Then the system $Ax > 0$, $x \geqq 0$ has solutions if and only if $Z(x)$ is conditionally definite.

H. W. Kuhn (London)

317:

Huyberechts, Simone. Sur le problème de l'unicité de la solution des jeux sur le carré-unité. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 200–216.

Two criteria are proposed to choose among optimal mixed strategies (F, G) in a game on the unit square with payoff $K(x, y)$, when the solution is not unique. The first advises the exclusive use of admissible optimal strategies. (A strategy G is admissible if $K(x, G) \geqq K(x, G')$ for all x implies $K(x, G) = K(x, G')$ for all x.) The second criterion suggests that the players adopt optimal mixed strategies (F, G) which minimize some measure of the dispersion of $K(x, y)$.

Relative to the first criterion, it is proved that if $K(x, y)$ is continuous and convex in y for each x then the optimal pure strategies for the second player form a complete class.

H. W. Kuhn (London)

318:

Rogers, C. A.; and Shephard, G. C. Convex bodies associated with a given convex body. J. London Math. Soc. **33** (1958), 270–281.

Let K be an n-dimensional convex body with volume $V(K)$. Then symmetrical convex bodies associated with K are defined as follows. The "difference body" $DK = K - K$. The "reflexion body" R_aK for each $a \in K$ is the minimal convex body containing K which is centrally symmetric in a; and if a is any point such that R_aK has minimum volume, then the body R_aK is designated RK. The associated $(n+1)$-dimensional body CK is the convex hull of the two sets determined by $(X_1, X_2, \cdots, X_n, 0) = (X, 0)$ and $(-X, 1)$ for $X \in K$. This paper establishes inequalities among the volumes $V(K)$ and the associated convex bodies of K. Theorems 2 and 3 are stated as examples. Theorem 2: $V(K) \leqq V(CK) \leqq 2^n V(K)/(n+1)$, with equality on the left if and only if K is centrally symmetric, and with equality on the right if and only if K is an n-dimensional simplex. Theorem 3: If $a \in K$, then $V(K) \leqq V(R_aK) \leqq 2^n V(K)$, with equality on the left if and only if K is centrally symmetric

with a as centre, and with equality on the right if and only if K is a simplex with a as a vertex.

P. C. Hammer (Madison, Wis.)

319:

Firey, Wm. J. A note on a theorem of H. Knothe. Michigan Math. J. **6** (1959), 53–54.

This note extends a theorem of Knothe [same J. **4** (1957), 53–56; MR **18**, 757].

Let $C(u)$ be the supporting cylinder in the direction u of a convex body K (in E_3). Let $B(u)$ be the breadth of K in the direction u and let $L(u)$ be the perimeter of the cross-section of $C(u)$. Let $B(u_0)$ and $B(u_1)$ be maximum and minimum values of $B(u)$, respectively. Let $S(u)$ be the lateral area of $C(u)$ between the two support planes of K perpendicular to u.

Theorem 1: If $S(u)$ is constant, then K is of constant breadth. Let $S(K)$ be the surface area of K. Theorem 2: (Knothe) If $S(u)=S(K)$, then K is a sphere.

P. C. Hammer (Madison, Wis.)

320:

Melzak, Z. A. A property of convex pseudopolyhedra. Canad. Math. Bull. **2** (1959), 31–32.

The author shows that there exists a convergent sequence of points in real three-dimensional Euclidean space whose convex hull (cover) is such that amongst all its plane sections there occurs a triangle similar to any preassigned triangle.

H. G. Eggleston (London)

GENERAL TOPOLOGY, POINT SET THEORY

See also 59, 96, 100, 131, 180.

321:

Ball, B. J. The sum of two solid horned spheres. Ann. of Math. (2) **69** (1959), 253–257.

If M is an Alexander horned sphere whose interior, U, is not simply connected, then $M+U$ is called a solid horned sphere. Bing [same Ann. **56** (1952), 354–362; MR **14**, 192] has shown that a continuum is homeomorphic with S^3 if it is the sum of three mutually exclusive sets M, U^1, U^2, such that (a) there is a homeomorphism of $M+U^i$ $(i=1, 2)$ into a solid horned sphere that carries M into the horned sphere, and (b) there is a homeomorphism of $M+U^1$ into $M+U^2$ that leaves each point of M fixed. Bing asked if condition (b) is superfluous. It is shown here that condition (b) can not be deleted in the theorem provided a more complicated surface, called a horned sphere of order 4, is substituted for the Alexander horned sphere.

W. R. Utz (Columbia, Mo.)

322:

Capel, C. E.; and Strother, W. L. Multi-valued functions and partial order. Portugal. Math. **17** (1958), 41–47.

Some results on partially ordered topological spaces are obtained, including a new proof (using L. E. Ward's order-theoretic characterization of a tree) of the following fixed-point theorem of A. D. Wallace [Bull. Amer. Math. Soc. **47** (1941), 757–760; MR **3**, 57]. If T is a tree, and if F is a weakly continuous (=upper semi-continuous) function from T to the closed connected subsets of T, then $x \in F(x)$ for some $x \in X$.

E. Michael (Seattle, Wash.)

323:

Lintz, Rubens G. La bi-cellule et les variétés dans un espace abstrait. Ann. Mat. Pura Appl. (4) **46** (1958), 343–348.

Based on concepts introduced in his earlier paper [same Ann. **43** (1957), 357–370; MR **19**, 437] the author introduces a concept of 2-cell, claimed to coincide, for Peano spaces, with that of L. Zippin [Amer. J. Math. **55** (1933), 207–217].

R. Arens (Los Angeles, Calif.)

324:

de Groot, J.; and Wille, R. J. Rigid continua and topological group-pictures. Arch. Math. **9** (1958), 441–446.

A topological space X is said to be a "topological group-picture" of a given group G if the group of homeomorphisms of X onto X is isomorphic to G. The authors show that any countable group admits a Peano continuum as a topological group picture. Some related problems are also studied.

H. H. Corson (Seattle, Wash.)

325:

Fort, M. K., Jr. ε-mappings of a disc onto a torus. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 51–54.

Let X and Y be metric spaces and $\varepsilon>0$ a real number. An ε-mapping of X onto Y is a mapping (i.e. continuous function) f of X onto Y such that for each y in Y the set $f^{-1}(y)$ has diameter less than ε. The author proves that for ε sufficiently small there exists no ε-mapping of the closed unit disk onto the torus. This answers a question proposed by S. Ulam in *The Scottish Book* [Lwów, 1935–1941, translated by S. Ulam, Los Alamos, 1957]. The same result has been found independently and by means of a different method by T. Ganea, and will appear in Fundamenta Mathematicae.

D. W. Hall (Endicott, N.Y.)

326:

Tsujimoto, Hitoshi. On quasi-compact mappings. Bull. Univ. Osaka Prefecture Ser. A **6** (1958), 19–24.

This is a detailed study of quasi-compactness of a mapping $f(A)=B$ on A and on inverse sets E in A, where A and B are topological spaces. The results represent extensions and completions of earlier ones of the reviewer, Rhoda Manning and others. For example it is shown that: (i) f is continuous and quasi-compact on A if and only if $f(E_*)=\overline{f(E)}$ for any inverse set E in A (E_* is the least closed inverse set in A containing E); (ii) f is continuous and quasi-compact on every inverse set in A if and only if $f(\bar{E})=\overline{f(E)}$ for every such set E; and (iii) both f and f^{-1} are continuous if and only if $f^{-1}[\overline{f(E)}]=\bar{E}$ for any inverse set E in A (continuity of f^{-1} means that $f^{-1}(\bar{Z})\subset\overline{f^{-1}(Z)}$ for any subset Z of B).

G. T. Whyburn (Charlottesville, Va.)

327:

Bourgin, D. G. A real continuous function on a space admitting two periodic homeomorphisms. Michigan Math. J. **5** (1958), 247–251.

In this paper there is established an invariance result concerned with even powers of two periodic fixed-point-free homeomorphisms R_1 and R_2 of (minimum) even periods m_1 and m_2 operating on a unicoherent locally connected compactum P. It is shown that if P admits a measure, if R_1 and R_2 are measure-preserving and if all

powers of R_i less than or equal to m_i are fixed-point-free, then for any real-valued mapping f on P there is a point in P whose images under f, fR_1 and fR_2 coincide. The existence of certain minimal invariant sets under the cyclic group generated by such homeomorphisms R is obtained as a consequence of this theorem.

G. T. Whyburn (Charlottesville, Va.)

ALGEBRAIC TOPOLOGY

See also 652.

328:

Weier, Joseph. Ueber eine Verschlingungsinvariante. Collect. Math. **10** (1958), 45–58.

Let P be a connected orientable closed 3-manifold, Q a connected orientable closed 2-manifold, $a \neq b$ two points of Q, and $f: P \to Q$ a map such that $f^{-1}(a)$ is the union $\bigcup A_i$ of finitely many mutually disjoint closed polygons A_i, and similarly $f^{-1}(b) = \bigcup B_i$. By choosing orientations we may associate with $f^{-1}(b)$ a 1-cycle $\sum \beta_i y_i$ where y_i is an orientation of B_i and β_i is the "degree" of f on B_i (in the sense utilized by Hopf for maps $S^3 \to S^2$). The A_i may be put into classes by declaring A_j and A_k to be equivalent if there is a path in P, starting on A_j and ending on A_k whose image is nullhomologous in Q. If A_1', A_2', $\cdots$, run through a class we may associate with them a 1-cycle $\sum \alpha_i x_i$, where x_i is an orientation of A_i' and α_i the degree of f on A_i'. This latter 1-cycle is called a solution-cycle of (f, a). If $z_1, z_2, \cdots$ is the set of bounding solution-cycles of (f, a), the author defines the map f to be "linked" if and only if $\sum z_i$ is linked with $\sum \beta_i y_i$. Thus if P, Q are spheres (so that all A_i are in the same class) f is linked if and only if its Hopf invariant is non-null.

The author's main result is that the property of being linked is an invariant of the homotopy class of maps $P \to Q$. This is proved by first generalizing to the situation in which the dimensions of P and Q are subject only to the condition $\dim P - \dim Q = 1$ and then studying "cylinder homotopies" from P to Q. A cylinder homotopy is a pair of homotopies f^τ, g^τ: $P \to Q$ such that, if A^τ is the coincidence set of f^τ and g^τ, then $\dim A^\tau \leq 1$ and there is a homotopy $t^\tau: A^0 \to P$ with $t^0 = 1$, $t^\tau(1, 1)$ onto A^τ if $\tau < 1$, and $t^1(A^0) \subseteq A^1$. The author defines geometric and algebraic "solution-classes" of (f, g) in a fairly obvious way and proves invariance theorems under cylinder homotopy. The main theorem results by taking for g^τ the constant homotopy at a fixed point of Q.

P. J. Hilton (Ithaca, N.Y.)

329:

Weier, Joseph. Ueber Homologieklassen gewisser topologischer Loesungen. Collect. Math. **10** (1958), 59–68.

In this paper the author continues the investigation of linking started in the paper reviewed above. The main result constitutes a sharpening of the main theorem of the earlier paper. Conserving terminology, let $\zeta_1, \zeta_2, \cdots$, be the set of bounding solution-cycles of (f, a) and let $z_1, z_2, \cdots$, be the set of solution-cycles of (f, b). The "link-type" of f then consists of those integers γ such that there exists a 2-chain w on P such that $\partial w = \sum \zeta_i$ and the intersection number $(w, \sum z_i)$ is γ. The author proves that the link-type is a homotopy invariant; that is, that it depends only on the homotopy class of f and not on the particular choice of map f nor on the reference points a, b. This theorem thus generalizes Hopf's theorem on maps $S^3 \to S^2$; like the latter it extends to maps of $(2n-1)$-manifolds into n-manifolds.

P. J. Hilton (Ithaca, N.Y.)

330:

Fadell, Edward. On fiber spaces. Trans. Amer. Math. Soc. **90** (1959), 1–14.

The first part of the paper is devoted to fiber spaces in the sense of Hurewicz [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961; MR **17**, 519]. Using the technique of "lifting functions" ten propositions are proved. For instance: Let (X, B, p) be a (Hurewicz) fiber space with B 0-connected and weakly locally contractible (i.e. each $b \in B$ admits a neighborhood contractible in B to b). Then there is a space F and an open covering $\{U_\alpha\}$ of B such that $(p^{-1}(U_\alpha), U_\alpha, p)$ is fiber-homotopy equivalent to $U_\alpha \times F$.

In the second part of the paper a new type of fiber space is introduced and explored. By definition (X, B, p) is required to verify the assertion of the above proposition. Clearly, the notion is invariant under fiber-homotopy equivalence. For these fiber spaces the author proves a form of covering homotopy theorem and a cross-section theorem.

S. Mardešić (Zagreb)

331:

Dold, Albrecht; and Lashof, Richard. Principal quasi-fibrations and fibre homotopy equivalence of bundles. Illinois J. Math. **3** (1959), 285–305.

Let H be a topological space with a continuous multiplication which is associative and has a two-sided unit. In analogy to the case of a topological group, the authors construct in this paper a universal quasi-fibration with fibre H. As an application, they obtain a classification of fibre bundles with respect to fibre homotopy equivalence.

Sze-tsen Hu (Palo Alto, Calif.)

332:

Lashof, R. K.; and Smale, S. Self-intersections of immersed manifolds. J. Math. Mech. **8** (1959), 143–157.

Une immersion f: $M^k \to X^{k+r}$ (M^k, X^{k+r} variétés C^∞) est dite n-normale si pour chaque n-uple de points distincts $x_1, x_2, \cdots, x_n$ de M, d'image commune, les images $\bar{M}_{x_i}$ des espaces tangents M_{x_i} ont une intersection de dimension $k-(n-1)r$ (intersection minimum). Les auteurs prouvent d'abord que cette propriété est générique (au sens de Thom) pour les immersions de variétés compactes. Soit Σ l'ensemble des points de M, compacte, qui ont une image commune par f avec au moins $n-1$ autres points de M: c'est alors une variété compacte immergée dans M.

Le but du mémoire est de prouver la formule:

$$\Sigma^* = \pm(\gamma(f) - \lambda W^r)^{n-1}$$

dans $H_{k-(n-1)r}(M)$ (qui généralise une formule de Whitney qui supposait $n=2$, M, X orientables, X compact).

Dans cette formule, Σ^* est la classe d'homologie définie par Σ, $W^r \in H^r(M)$ la classe de Stiefel-Whitney du fibré normal à M dans X, λ la dualité de Poincaré, et $\gamma(f)$ l'image du générateur choisi de $H_k(M)$ par:

$$H_k(M) \xrightarrow{f} H_k(X) \xrightarrow{\lambda} H^r(X) \xrightarrow{f^*} H^r(M) \xrightarrow{\lambda} H_{k-r}(M).$$

Le produit est le dual du cup-produit, les coefficients

sont Z ou Z_2 suivant l'orientabilité et r, le signe $\pm$ dépend de k, r, n.

Quelques applications sont données, en particulier: Si $f: M^k \to X^{k+r}$ est 2-normale et au plus 2-1, alors $f(\Sigma)$ est non-orientable si r est impair, orientable si r est pair.

Si k est pair, il existe au voisinage de toute immersion de M^k dans $E^{2k-1} \subset E^{2k}$ un plongement de M^k dans E^{2k}.

R. Deheuvels (Bourg-La-Reine)

333:

Weier, Josef. Eine topologische Konstante. Monatsh. Math. **63** (1959), 112–123.

The author considers a pair of mappings $f(p)$, $g(p)$ from an m-dimensionable orientable closed euclidean manifold M in E_r into an n-dimensional orientable closed manifold N in E_s. He decomposes the set A of $p \in M$ where $f(p)=g(p)$ into equivalence classes by defining a and b as equivalent relative to (f, g) when a closed curve obtained by combining the f and g images of a curve on M joining a and b is homologous to zero on N. The set A and the equivalence classes A_i are compact and the latter are finite in number. It is then shown that a transformation ψ can be defined from the set of equivalence classes into the $m-n$ dimensional integral Betti group B of N in such a way that the number ζ' of equivalence classes which are mapped into elements of B other than the neutral element has special significance, namely, for any two mappings f' and g' of M into N which are homotopic to f and g, respectively, the equation $f'(p)=g'(p)$ has at least ζ' solutions.

G. T. Whyburn (Charlottesville, Va.)

334:

Weier, Joseph. Sur des matrices de torsion caractéristiques de M. G. de Rham. C. R. Acad. Sci. Paris **248** (1959), 1752–1754.

The author takes as his point of departure the matrices of linking coefficients indicated in his title. He defines further matrices, but does not appear to state any results.

J. F. Adams (Cambridge, England)

335:

Harary, Frank. The number of oriented graphs. Michigan Math. J. **4** (1957), 221–224.

In this paper the author extends his investigation of the number of linear, directed, rooted and connected graphs [Trans. Amer. Math. Soc. **78** (1955), 445–463; MR **16**, 844] to oriented graphs. An oriented graph is a graph in which each edge has a definite direction assigned to it, and in which two nodes are joined by at most one edge. In this the definition of oriented graphs differs from that of directed graphs, because in the latter two nodes can be joined by two edges, one in each direction. Using the well-known results of G. Pólya [Acta. Math. **68** (1937), 145–254], the author obtains a formula for a counting polynomial for the number of different oriented graphs of given order.

G. A. Dirac (Hamburg)

336:

Tutte, W. T. A homotopy theorem for matroids. I, II. Trans. Amer. Math. Soc. **88** (1958), 144–174.

A matroid is a family M of subsets of a finite set, no one of which contains another, satisfying the following condition: For any X, Y, elements of M, a, b, elements of X, $a \in Y$, $b \notin Y$, there is $Z \in M$ such that

$$b \in Z \subset X \cup Y - \{a\}.$$

The principal example is the family of sets of k-simplexes in a complex which are minimal supports for cycles. In more detail, let R be either the ring of integers or the field of integers mod p; let N be any subgroup of the additive group of all functions on some finite set to R. The author shows that the minimal supports of functions in N constitute a matroid $M(N)$.

The chain group N, and the associated matroid $M(N)$, are called binary if R is the two-element field, regular if R is the integers and for every nonzero $g \in N$ whose support is minimal there is $g_0 \in N$ having the same support and having all its coefficients 0, 1, or -1. The principal results of the present papers are characterizations of binary and of regular matroids.

To this end the author investigates the abstract configuration (a semilattice) consisting of all unions of elements of a matroid M. This semilattice is shown to satisfy the Jordan-Dedekind chain condition (i.e., maximal chains with the same ends have the same length), so that it admits a dimension function with the usual properties. Accordingly, the elements of M are called "points" the unions "flats", and the terms "line", "plane" are similarly employed. A flat F is called "disconnected" if it is a union of disjoint non-null subsets F', F'', such that every point in F is a subset of F' or of F''.

The binary matroids are characterized by the condition that every connected line contains exactly three points. The regular matroids are binary, and are characterized by the further condition that the associated configuration contains neither the Fano configuration nor a certain related three-dimensional configuration.

The second paper gives the characterizations. The first paper establishes the homotopy theorem of the title, which is needed for the regular matroids. The nature of the theorem is that certain elementary operations suffice to reduce any "closed path" in a matroid to a point, while remaining in the complement of a specified "convex" class of points.

J. Isbell (Seattle, Wash.)

337:

Tutte, W. T. Matroids and graphs. Trans. Amer. Math. Soc. **90** (1959), 527–552.

"In the original paper on matroids, Hassler Whitney pointed out [Amer. J. Math. **57** (1935), 507–533] that the circuits of any finite graph G define a matroid. We call this the circuit-matroid and its dual the bond-matroid of G. In the present paper we determine a necessary and sufficient condition, in terms of matroid structure, for a given matroid M to be graphic (cographic), that is the bond-matroid (circuit-matroid) of some finite graph. The condition is that M shall be regular and shall not contain, in a sense to be explained, the circuit-matroid (bond-matroid) of a Kuratowski graph." (Author's introduction.)

The paper begins with the definition and basic properties of dual matroids. Using much of the theory previously developed by the author [see the article reviewed above] he goes on to define minors of matroids. The minors of a graphic matroid apparently correspond one-one with the subgraphs; at any rate, "contain" in the main theorem means "contain as a minor". The author proves (after the main theorem) that if M_0 is a connected matroid of subsets of a set A and no two points of A lie in exactly the same members of M_0, then a matroid M contains M_0 as a minor if and only if the configuration associated with M_0 can be embedded by a dimension-preserving semilattice isomorphism in the configuration associated with M.

J. Isbell (Seattle, Wash.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 33, 77, 93, 97, 133, 149, 229.

338:

Vincze, I. Eine Bemerkung zur Differentialgeometrie der Flächen. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. 1 (1958), 71–72.

The classical theorem of Beltrami-Enneper says that at a hyperbolic point of an analytic surface the torsion of an asymptotic line determines the Gaussian curvature. Here it is pointed out by two examples that the asymptotic lines do not determine the mean curvature. However, if $1/R_n$ is the curvature of a normal section at P in a plane perpendicular to an asymptotic line, then the mean curvature is $1/2R_n$. *A. Schwartz* (New York, N.Y.)

339:

Soós, Gy. Über die geodätischen Abbildungen von Riemannschen Räumen auf projektiv-symmetrische Riemannsche Räume. Acta Math. Acad. Sci. Hungar. **9** (1958), 359–361.

N. S. Senyukow has extended a classical theorem of Beltrami and proved that a riemannian space which admits a non-trivial geodesic mapping (locally) on a symmetric riemannian space is necessarily of constant curvature [Dokl. Akad. Nauk SSSR **98** (1954), 21–23; MR **16**, 515].

In the present paper the theorem is extended still further, the symmetric space being replaced by a projective-symmetric space, i.e. a riemannian space in which the covariant derivative of the Weyl tensor $W^h{}_{ijk}$ vanishes at every point. *A. G. Walker* (Liverpool)

340:

Marcus, Froim. Sur les surfaces dont les courbes asymptotiques d'un système sont centro-affine équivalentes. Bul. Inst. Politehn. Iaşi (N.S.) 4 (8) (1958), 39–44. (Romanian. Russian and French summaries)

O. Mayer [Ann. Sci. Univ. Jassy **21** (1934), 1–77] established centro-affine differential geometry. Here the author determines the surfaces on which the members of a family of asymptotic curves are centro-affinely equivalent to each other. These surfaces are all minimal centro-affine surfaces, and in each case the asymptotic curves of the second family are also centro-affinely equivalent to each other. Chenkuo Pa worked on the same problem in projective differential geometry [Univ. Nac. Tucumán Revista A **3** (1942), 341–349; MR **5**, 108].

A. Schwartz (New York, N.Y.)

341:

Vincensini, Paul. Sur certaines équations aux dérivées partielles du deuxième ordre. Ann. Sci. École Norm. Sup. (3) **75** (1958), 153–166.

Let O be a fixed point and D any line of Euclidean three-space. Let Δ be the line through O parallel to D. Then line transformation $T(O,\alpha)$ carries D into the line D' obtained by rotating D about Δ through angle α. The $T(O,\alpha)$ with O fixed constitute a group, and the author first determines some of the quantities left invariant when the $T(O,\alpha)$ operate on a line congruence (D). Second, if (D) is normal, if Θ is a surface orthogonal to the rays of (D), if I is the projection of O on line D of (D), if M is the point at which D pierces Θ, and if $\varphi=\overline{IM}$, then the invariants previously determined can be written simply in terms of $\Delta\varphi$, $\Delta_2\varphi$, and $\Delta_{22}\varphi$, where Δ, Δ_2, Δ_{22} are Beltrami differential parameters relative to the linear element of the spherical image of (D) [see the author's paper Ann. Sci. École Norm. Sup. (3) **48** (1931), 397–438]. Thus the partial differential equations for many normal congruences with given focal properties can be written down in invariant form quickly, and four interesting examples are given. Third, a partial differential equation of the form $F(\Delta_{22}\varphi, \Delta\varphi)=0$, F any function, makes a statement about certain invariants of a normal congruence (D). $T(O,\pi/2)$ enables one to consider instead a special normal congruence (D'), with mean envelope surface just a point, for which the same statement on invariants can be made. If the congruences (D') are known, the solution φ can be constructed by integrating an exact differential. Fourth, to illustrate a case where this method can be used, the author integrates completely the equations $\Delta\varphi-\Delta_{22}\varphi=1$ corresponding to the different possible forms of the linear element of the unit sphere.

A. Schwartz (New York, N.Y.)

342:

Ginatempo, Nicola. Alcune congruenze di Bianchi. Boll. Un. Mat. Ital. (3) **13** (1958), 355–359. (English summary)

The author takes up earlier work of Bianchi and Tortorici on W-congruences having a prescribed focal surface, and characterises those associated with an infinitesimal motion. *T. J. Willmore* (Liverpool)

343:

Nagata, Yukiyoshi. Remarks on the normal curvature of a vector field. Tensor (N.S.) **8** (1958), 177–183.

Let X_m represent an m-dimensional subspace of a metric space X_n, C a curve of X_m, and suppose that there is defined a unit vector field v^λ $(\lambda=1,\cdots,n)$ over and tangential to X_m. Denote by ${}_v\kappa_n$ the magnitude of the component normal to X_m of the covariant derivative of v^λ with respect to X_n, taken along C. Regarding X_n as a Riemannian space, T. K. Pan [Amer. J. Math. **74** (1952), 955–966; MR **14**, 406] called ${}_v\kappa_n$ the normal curvature of the field v^λ with respect to C. In a previous paper the author [Tensor (N.S.) **5** (1955), 17–22; MR **17**, 190] extended the results of Pan concerning this concept, regarding X_n as a Finsler space, and taking the element of support as the unique unit normal of X_m, where $m=n-1$. In the present paper the element of support is taken tangentially to X_m (with $m<n$), so that inevitably certain differences emerge. When v^λ is tangential to C and coincident with the element of support, ${}_v\kappa_n$ is identical with the normal curvature of C. Asymptotic directions are defined by means of directions for which ${}_v\kappa_n$ vanishes, and their properties are discussed in some detail.

H. Rund (Durban)

344:

Nagano, Tadashi. Sur des hypersurfaces et quelques groupes d'isométries d'un espace riemannien. Tôhoku Math. J. (2) **10** (1958), 242–252.

The author studies a Killing vector field orthogonal to a submanifold B (particularly, a hypersurface) of a Riemannian manifold M. The first theorem states that if a hypersurface B is complete and if a connected group G of isometries of M is orthogonal to B (i.e., for every

$b \in B$, the orbit $G(b)$ is orthogonal to B at b), then (1) M is complete, (2) B is totally geodesic, (3) $M = G(B)$ and (4) $\dim G = 1$. The second theorem (with some modification in order to simplify the statement): If B is a closed submanifold of M and if $G(b)$ is the orthogonal complement to B at every $b \in B$, then M is a fibre bundle over G/H with fibre B and group H, where H consists of those elements of G which leave B invariant. (Each fibre is totally geodesic and the associated principal fibre bundle is $(G, G/H, H)$). It is proved in the last section that if B is a compact hypersurface in a compact M and if G is a 1-parameter group of isometries of M which is orthogonal to B and leaves no point of B (consequently, no point of M) fixed, then the group $I(M)$ of isometries of M is isomorphic to $G/H \times I_B(M)$, where $I_B(M)$ consists of all isometries of M which leave B invariant. If, moreover, M is homogeneous, then M is locally a direct product (as a Riemannian space) of B with the real line and the group operation of G is an obvious translation.

S. Kobayashi (Cambridge, Mass.)

345:

Yano, Kentaro; and Nagano, Tadashi. Einstein spaces admitting a one-parameter group of conformal transformations. Ann. of Math. (2) **69** (1959), 451–461.

A Riemannian manifold is called an Einstein space if the Ricci tensor is proportional to the metric tensor with a constant ratio. A transformation f of a Riemannian manifold M is said to be conformal if $f^*(ds^2) = \sigma \cdot ds^2$, where σ is a positive function on M. If σ happens to be a constant, f is called a homothetic transformation. The authors prove the following theorem: if a complete Einstein space M of dimension >2 admits a 1-parameter group of non-homothetic conformal transformations, then it is isometric to a sphere. Let g_t be the given 1-parameter group and let $g_t^*(ds^2) = \sigma_t \cdot ds^2$. The function $\varphi = (\partial\sigma_t/\partial t)_{t=0}$ plays an important role in the proof. The proof is divided into two cases: (A) The Ricci tensor is zero; (B) the Ricci tensor is not zero. The authors eliminate (A) by proving that $d\varphi$ is a parallel field. In the case (B), they prove that there exists a unique pair of points ξ_+ and ξ_- such $\varphi(\xi_+) = 1$ and $\varphi(\xi_-) = -1$ and then construct an isometry of M onto a sphere which maps ξ_+ and ξ_- into the north and south poles, respectively. *S. Kobayashi* (Cambridge, Mass.)

346:

Akbar-Zadeh, Hassan. Sur une connexion coaffine d'espace d'éléments linéaires. C. R. Acad. Sci. Paris **247** (1958), 1707–1710.

L'auteur étend à un espace d'éléments linéaires des résultats établis par Madame Maurer-Tison [mêmes C. R. **246** (1958), 240–243; MR **19**, 1141] sur les géodésiques et formes de courbure de deux connexions (linéaire et coaffine) canoniquement associées sur une variété différentiable. *J. Renaudie* (Rennes)

347:

Miron, R. La courbure et la torsion de parallélisme dans la géométrie des variétés non holonomes. An. Şti. Univ. "Al. I. Cuza" Iaşi. Sect. I (N.S.) **3** (1957), 171–181. (Romanian. Russian and French summaries)

The notion of curvature of parallelism has been introduced by O. Mayer (1924) for surfaces in the euclidean 3-space (E_3) and by E. Bortolotti (1925) for subspaces of riemannian spaces. The latter has defined also curvatures of successive orders which in the case of surfaces in a 3-dimensional space can appropriately be called curvature and torsion of parallelism.

In the present paper the author extends the above notions to a 'regular' system of planes given along a curve. The results are then applied to the anholonomic submanifolds E_3^2 of E_3 and generalizations of formulae, found by O. Mayer and E. Bortolotti, are thus obtained.

R. Blum (Saskatoon, Sask.)

348:

Miron, Radu. Observations sur certaines formules de la géométrie des variétés non holonomes E_3^2. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), 19–24. (Romanian. Russian and French summaries)

The author derives in this note formulae for the normal curvature $(1/R)$ and geodesic torsion $(1/T)$ of the following curves in an anholonomic E_3^2 of the euclidean S_3: (1) asymptotic lines; (2) lines of extremal normal curvature; (3) and (4) lines of extremal geodesic curvature and torsion. The results are then interpreted as points on a circle in the $(1/R, 1/T)$ plane.

R. Blum (Saskatoon, Sask.)

349:

Gołąb, S. Géométrie différentielle vis-à-vis des hypothèses d'une faible régularité. Rev. Math. Pures Appl. **1** (1956), no. 3, 99–112.

This is an address reviewing progress differential geometers have made in finding the lower limits of regularity on surfaces under which classical theorems hold. Despite the 114 items in the bibliography (which, of course, doesn't pretend to completeness) many of the names mentioned in the text do not occur.

L. W. Green (Minneapolis, Minn.)

PROBABILITY

See also 110, 114, 412, 416, 448, 576, 577, 644.

350:

Hermes, Hans. Zum Einfachheitsprinzip in der Wahrscheinlichkeitsrechnung. Dialectica **12** (1958), 317–331. (French and English summaries)

"Shimony, Lehman and Kemeny recently developed a foundation of the theory of confirmation by reduction to the concept of rational betting. This procedure yields essentially only the axioms first stated by Kolmogoroff. It is well known that these are not sufficient for application. Thus it is necessary to search for a new principle if one wants to motivate new axioms. This can be done by a principle of simplicity, which expresses that the probability of a hypothesis increases with its degree of simplicity. A critical survey is given of several attempts which have been tried in Münster, especially by Kiesow and W. Oberschelp, to make the notion of simplicity precise. The simplicity is reduced to syntactical properties of propositions." (Author's summary.)

L. J. Savage (Cannes)

351:

Kallianpur, Gopinath. A note on perfect probability. Ann. Math. Statist. **30** (1959), 169–172.

Consider the field $\mathfrak{G}$ of linear sets that are measurable

relative to each complete Stieltjes measure on the line. Given a measurable space with a separable sigma-field $\mathfrak{F}$ of measurable sets, such that the range of each $\mathfrak{F}$ measurable function f (from the space to the line) is measurable $\mathfrak{G}$, it is shown that each probability measure space built over a separable sigma-subfield of $\mathfrak{F}$ is perfect in the sense of Gnedenko and Kolmogorov [*Limit distributions for sums of independent random variables*, transl. by K. L. Chung, Addison-Wesley, Cambridge, Mass., 1954; MR **16**, 52]; the converse is also true. D. Blackwell's class of measurable spaces (for which the range of each f is analytic) [Proc. 3rd Berkeley Sympos. on Math. Statist. and Prob., 1954–1955, vol. 2, pp. 1–6, Univ. of Calif. Press, Berkeley–Los Angeles, 1956; MR **18**, 940] is included in Kallianpur's class, but it is not known whether the latter is, in fact, more extensive.

H. P. McKean, Jr. (Cambridge, Mass.)

352:

Rihter, Vol'fgang. The limit behaviour of the χ^2 distribution in the case of large deviations. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 652–654. (Russian)

Let $\chi^2=\sum_{j=1}^{s+1}(\nu_j-np_j)^2/(np_j)$, where the random vector $(\nu_1,\cdots,\nu_{s+1})$ has a multinomial distribution with parameters $n, p_1,\cdots,p_{s+1}$ (all positive). An asymptotic expression for $P\{\chi^2>\tau^2\}$ is given for $\tau=o(n^{1/2})$ as $n\to\infty$. For $\tau=o(n^{1/6})$ it reduces to the classical approximation, but for larger values of τ it explicitly depends on the p_j.

W. Hoeffding (Chapel Hill, N.C.)

353:

Lukacs, Eugene. Some extensions of a theorem of Marcinkiewicz. Pacific J. Math. **8** (1958), 487–501.

The author is concerned with conditions under which a function $f(t)$ can or cannot be the characteristic function (ch.f.) of a (probability) distribution, and he obtains by fairly elementary arguments the following results. Theorem 1: Let $P(t)=\sum_{\nu=0}^{m}c_\nu t^\nu$ be a polynomial of degree m, and let

$$f_1(t)=\kappa_1\exp P(t),$$
$$\kappa_1=\exp(-c_0),\cdots,f_n(t)=\kappa_n\exp[f_{n-1}(t)/\kappa_{n-1}],$$
$$\kappa_n=\exp(-1/\kappa_{n-1}).$$

Then $f_n(t)$ cannot be a ch.f. if $m>2$. Theorem 2: With $P_m(t)$ as in theorem 1, the function

$$f(t)=\exp[\lambda_1(e^{it}-1)+\lambda_2(e^{-it}-1)+P_m(t)]$$

is a ch.f. if and only if $\lambda_1\geqq 0$, $\lambda_2\geqq 0$, $m\leqq 2$, and $P_2(t)=a_1it-a_2t^2$, where a_1 and a_2 are real and $a_2\geqq 0$. In connexion with theorem 1 he observes that if $m\leqq 2$, $P_m(t)=a_1it-a_2t^2$, where a_1 and a_2 are real and $a_2\geqq 0$, then $f_n(t)$ is a ch.f. $(n=1,2,\cdots)$. The special case of theorem 1 with $n=1$ and that of theorem 2 with $\lambda_1=\lambda_2=0$ are identical with a result due to J. Marcinkiewicz [Math. Z. **44** (1938), 612–618], who deduced it from the following more general theorem: an entire function of finite order $\rho>2$ whose exponent of convergence ρ_1 is less than ρ cannot be a ch.f. By bringing in some results from the theory of entire functions the author obtains a new proof of this last theorem.

H. P. Mulholland (Exeter)

354:

Medgyessy, P. Partial integro-differential equations for stable density functions and their applications. Publ. Math. Debrecen **5** (1958), 288–293.

In an earlier paper [Magyar Tud. Akad. Mat. Kutató Int. Közl. **1** (1956), 489–518 (1957); MR **20** #1345] the author obtained a linear partial differential equation for the density function of a stable distribution with rational exponent $\alpha\neq 1$. In the present paper he considers stable distributions with (not necessarily rational) exponent $\alpha\neq 1$ and derives two partial integro-differential equations for their frequency functions. One of these equations holds if $0<\alpha<1$, the other if $1<\alpha<2$. These equations can be used to analyse a mixture of stable distributions with common exponent.

E. Lukacs (Washington, D.C.)

355:

Kolmogorov, A. Sur les propriétés des fonctions de concentrations de M. P. Lévy. Ann. Inst. H. Poincaré **16** (1958), 27–34.

Let $\xi_1,\cdots,\xi_n$ be mutually independent random variables with sum ξ, and let Q_k be the Lévy [Théorie de l'addition des variables aléatoires, Paris (1937)] function of concentration of ξ_k, Q that of ξ. The following generalization of a theorem in the above reference is proved. There is a constant C such that the inequalities

$$L\geqq l,\quad L^2\geqq l^2\log_2\left\{\sum_1^n[1-Q_k(l)]\right\}$$

imply that $Q(L)\leqq CL/(ls^{\frac{1}{2}})$.

J. L. Doob (Urbana, Ill.)

356:

Kuiper, Nicolaas H. Alternative proof of a theorem of Birnbaum and Pyke. Ann. Math. Statist. **30** (1959), 251–252.

Let $U_1, U_2,\cdots,U_n$ be the ordered values of n independent random variables, each of which is uniformly distributed over (0, 1). Let i^*, U^* be defined (uniquely with probability 1) by $\max i/n-U_1=i^*/n-U_{i^*}$, $U^*=U_{i^*}$. Then U is uniformly distributed over (0, 1). This was proved by Birnbaum and Pyke by combinatorial techniques, [same Ann. **29** (1958), 179–187; MR **20** #393]. A shorter proof was given by this reviewer [ibid. **29** (1958), 188–191; MR **20** #2051]. Kuiper now presents a very short proof based on a clever invariance device which should find other uses.

M. Dwass (Evanston, Ill.)

357:

Herzel, Amato. Influenza del raggruppamento in classi sulla probabilità e sull'intensità di transvariazione. Metron **19** (1958), no. 1–2, 199–242.

Let X_1, X_2 be independent random variables with the medians M_1, M_2, respectively. According to G. Gini, the event: sign $(X_1-X_2)=-\text{sign}(M_1-M_2)$ is called a transvariation. Consider $P=2p$, where p is the probability of this event. The author assumes that X_1, X_2 are discrete random variables, with the same sets of equidistant values, and investigates the effect on P of grouping these values into classes with larger class-intervals of equal length. A similar study is carried out for the effect of grouping on the intensity of transvariation, an index related to the probability of the event: sign $(X_1-X_2)=-\text{sign}(m_1-m_2)$, where m_1, m_2 are the population means of X_1, X_2.

Z. W. Birnbaum (Seattle, Wash.)

358:

Fisz, M. A limit theorem for non-decreasing random functions. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 485–487.

The Glivenko-Cantelli theorem can be extended as

follows. Let $\{Y_k(t), -\infty<t<\infty\}$ $(k=1, 2, \cdots)$ be a sequence of real, independent and identically distributed stochastic processes non-decreasing in the sense that for $t_1<t_2$, $P\{Y_k(t_1)\leq Y_k(t_2)\}=1$. Provided that all the processes and their sums considered are separable, and that $EY_k(t)=F(t)$ exists and is bounded, $n^{-1}\sum_{k=1}^{n} Y_k(t)$ converges uniformly almost surely to $F(t)$. [The assumption which the author makes of separability of each $Y_k(t)$ is not sufficient, since there exist examples of two separable, independent and identically distributed processes whose sum is not separable.] See Errata

T. S. Ferguson (Los Angeles, Calif.)

359:

Basharin, G. P. Multiplex limited number distribution of busy lines in the second cascade switchboard in a telephone system with refusals. Soviet Physics. Dokl. **121 (3)** (1958), 718–721 (280–283 Dokl. Akad. Nauk SSSR).

The multi-dimensional central limit theorem is used to compute an approximate distribution for the random vector whose i component is the number of calls on the ith switch in the second stage of a two-stage telephone switching network. The approximation is valid when the number of switches in the first stage, i.e., the size of the switches in the second stage, is large.

V. E. Beneš (Murray Hill, N.J.)

360:

Révész, P. On the limit distributions of sums of dependent random variables. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 135–142.

L'auteur utilise un résultat du A. N. Kolmogoroff pour la démonstration du théorème suivant: soit $\xi_1, \cdots, \xi_n, \cdots$ une suite de variables aléatoires non nécessairement indépendantes. Supposons qu'il existe des suites de nombres réels A_n, B_n $(\lim_{n\to\infty} B_n=\infty)$, une mesure μ et une fonction de répartition $F(x)$ telles que $\mu((\xi_1+\cdots+\xi_n)/B_n-A_n<x)$ converge en loi vers $F(x)$ lorsque $n\to\infty$. Alors il en est de même de $\nu((\xi_1+\cdots+\xi_n)/B_n-A_n<x)$ pourvu que la mesure ν soit absolument continue par rapport à la mesure μ.

A. Fuchs (Strasbourg)

361:

Beck, Anatole. Une loi forte des grands nombres dans des espaces de Banach uniformément convexes. Ann. Inst. H. Poincaré **16** (1958), 35–45.

Proof of a theorem announced in another paper [C. R. Acad. Sci. Paris **246** (1958), 696–698; MR **19**, 1202].

J. L. Doob (Urbana, Ill.)

362:

Mourier, Édith. Lois de probabilité conditionnelles; existence et détermination d'un résumé exhaustif pour la discrimination entre plusieurs lois de probabilité dans des espaces de Banach. C. R. Acad. Sci. Paris **247** (1958), 1552–1554.

Remarks on conditional probabilities, the sufficiency of the likelihood ratio statistic for the discrimination between two probability laws, and the value of the likelihood ratio for a pair of Gaussian processes u and $u+v$, where v is a fixed continuous function. U. Grenander computed the latter some time ago [Ark. Mat. **1** (1951), 195–277; MR **12**, 511].

H. P. McKean, Jr. (Cambridge, Mass.)

363:

Nisio, Makiko. On a new definition of stochastic integral by random Riemann sum. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 25–31.

Let $\{Y(t, \omega), 0\leq t\leq\infty\}$ be a Wiener process on the probability space (Ω, B, P) and, for $n=1, 2, \cdots$, $\{P_n(t, \omega'), 0\leq t<\infty\}$ a Poisson process on a probability space (Ω', B', P'), n being the mean value of $P_n(1, \omega')$, where ω' is independent of ω. (It makes no difference whether the processes P_n are mutually independent or not.) Let $t_i^n(\omega')$ be the ith jumping time of the process P_n. For a suitably measurable function $\Phi(t, \omega)$ whose second moment is integrable, the stochastic integral

$$(*) \qquad \int_0^\infty \Phi(t, \omega)\,dY_t(\omega)$$

is defined to be equal to $\phi^*(\omega)$ if and only if $\phi^*(\omega)$ does not depend on ω' and

$$\lim_{n\to\infty} P'\left[\int_\Omega (S_n(\omega, \omega')-\phi^*(\omega))^2 P(d\omega) > \varepsilon\right] = 0 \quad (\varepsilon > 0),$$

where

$$S_n(\omega, \omega') = \sum_{i\geq 1} \Phi(t_i^n(\omega'), \omega)[Y(t_{i+1}^n(\omega'), \omega)-Y(t_0^n(\omega'), \omega)].$$

It is shown that this integral exists and is uniquely determined a.e. (P), that it has some of the usual properties, and that it is equal a.e. (P) to the stochastic integral defined by K. Itô [Proc. Imp. Acad. Japan, **20** (1944), 519–524; MR **7**, 313].

W. Hoeffding (Chapel Hill, N.C.)

364:

Jiřina, Miloslav. Stochastic branching processes with continuous state space. Czechoslovak Math. J. **8 (83)** (1958), 292–313. (Russian summary)

The usual branching process with n types of objects is a Markov process whose states are n-dimensional vectors with non-negative integer coordinates, the ith coordinate representing the number of objects of type i. The author generalizes to include the case where the states are still non-negative n-dimensional vectors but the coordinates need not be integers. (This is to be distinguished from generalizations in a different direction, where there is a continuous range of types.) To illustrate with the temporally homogeneous case, let E be the non-negative orthant of Cartesian n-space, let $P_1, \cdots, P_n$ be infinitely divisible probability measures on E, and let

$$\int_E P_i(db)e^{b_1x_1+\cdots+b_nx_n} = e^{\phi_i(x)}.$$

If the state at the integer time t is $(a_1, \cdots, a_n)\in E$, then the state at time $t+1$ has the conditional moment-generating function $e^{a_1\phi_1(x)+\cdots+a_n\phi_n(x)}$. If $P(s, a, t, A)$ is the transition probability from state a at time s to the set A at time t, s and t integers, $s\leq t$, then processes constructed in the above way have the characteristic property

$$(*)\ P(s, a^1+a^2, t, A) = \iint_{b^1+b^2\in A} P(s, a^1, t, db^1)P(s, a^2, t, db^2).$$

It is shown that the basic results for integer-state branching processes (iteration of generating functions, extinction probabilities, role of final groups) have appropriate counterparts for the new processes. Next suppose t is continuous. Here the author considers only processes of

the pure jump type. To illustrate with the temporally homogeneous case in 1 dimension, let p be a probability measure on $[1, \infty)$. The branching process is a jump-type Markov process whose intensity of transition from the state x to the set $A \subset (x, \infty)$ is defined as $xp(A-x+1)$. It is necessary to rule out downward transitions in order to prevent the state from taking negative values, and the state never goes to 0 in such processes. It is shown that under suitable conditions every intensity of the above sort determines a Markov process on E whose transition probabilities satisfy (*) for $s \geqq 0$, $t \geqq s$.

T. E. Harris (Santa Monica, Calif.)

365:

Nelson, Edward. The adjoint Markoff process. Duke Math. J. **25** (1958), 671–690.

Let X be a locally compact Hausdorff space, let C be the space of continuous functions on X, vanishing at ∞, and let B be a linear space (with supremum norm) of bounded continuous functions on X, containing C. Let P be a linear transformation of B into itself, taking positive ($\geqq 0$) functions into positive functions, of norm $\leqq 1$, with $P1=1$ if X is compact, and suppose that for every positive f in B and every x in X, $\sup_n (P^n f)(x) > 0$. An existence theorem for linear functionals is proved implying that there is a regular measure m on X such that $mU > 0$ if U is open and not empty, and that $\int Pf\,dm \leqq \int f\,dm$ if f is positive and has compact support. A corresponding result for semigroups of transformations on B is also obtained.

Now suppose that X is any measurable space with the same structure as a Borel linear set. In the following, all measures are to be understood to be totally sigma finite. Let p be a substochastic transition function: $p(x, \cdot)$ is a measure for each x; $p(\cdot, E)$ is measurable for each E; $p \leqq 1$. If the integrals make sense, define

$$(Pf)(x) = \int f(y)p(x, dy), \quad (P^*\mu)(E) = \int p(x, E)d\mu(x).$$

If $P^*\mu \leqq \mu$, μ is called 'subinvariant', 'invariant' if there is equality. If p is substochastic, and if m is a subinvariant measure, it is shown that P is well-defined on $L^\alpha(m)$, and that P^*f can be defined for f in L^1 (and by extension to L^α) as the Radon-Nikodym derivative with respect to m of $P^*\mu$, where $d\mu = f dm$. There is then a simple adjointness relationship between P on L^α and P^* on L^β, where $1/\alpha + 1/\beta = 1$. Moreover P^*f is itself determined by a substochastic transition function p^*. If

$$\nu_x(E) = \sum_1^\infty 2^{-n} p^n(x, E),$$

where p^n is the nth iterate of the kernel p, and if (*)ν_x has the same null sets for every x, p and p^* are determined by transition densities which are a.e. transposes of each other.

Results on invariant measures are obtained which yield the following, with the help of a theorem of Harris [Proc. Third Berkeley Symposium Math. Statist. Probability, 1954–1955, vol. II, pp. 113–124, Univ. of California Press, Berkeley, 1956; MR **18**, 941]. If p is substochastic, if (*) is satisfied, if there is a point x_0 in X with $\sum_1^\infty p^n(x_0, E) = \infty$ whenever $\nu_x(E) > 0$, then there is an invariant measure m with the same null sets as ν_x, and such that every subinvariant measure is a multiple of m.

If X is a differentiable manifold, and if there is a semigroup with infinitesimal generator a second order elliptic differential operator acting on functions on X, it is shown as an application that there is a unique invariant probability measure, and the adjoint semigroup is identified.

J. L. Doob (Urbana, Ill.)

366:

Urbanik, K. Limit properties of Markoff processes. Rozprawy Mat. **13** (1957), 46 pp. (Polish. English and Russian summaries)

A Markov process is studied which has a stationary Markov transition function p and whose family Ω of sample functions consists of step functions ω from the non-negative half-line to a locally compact separable metric space X, all of which have a common value x_0 for $t=0$. Let P be the corresponding probability measure on the σ-algebra generated by the family of all subsets of Ω of the form $\{\omega : \omega(t) \in U\}$, where $t \geqq 0$ and U is an arbitrary open subset of X. A point x in X is called (a) a limit point or (b) a critical point of the process if (a) for every neighborhood U of x

$$\lim_{t\to\infty} P\Big(\bigcup_{T \geqq t} \{\omega : \omega(T) \in U\}\Big) > 0$$

or (b) for every $t \geqq 0$, $p(t, x, \{x\}) = 1$, respectively.

Under mild regularity assumptions the following theorems are proved. (1) A point x in X which is not a critical point of the process is its limit point if and only if for every neighborhood U of x, $\int_0^\infty p(t, x_0, U)dt = \infty$. (2) For almost all ω in Ω,

$$L(\omega) = \bigcap_{t \geqq 0} \overline{\bigcup_{T \geqq t} \{x : \omega(T) = x\}}$$

consists of limit points of the process. (3) There exists a sequence $\{\Gamma_i\}$ of disjoint open subsets of X which contains the interior of $L(\omega)$ for almost all $\omega \in \Omega$.

Several applications are given in which X is discrete. They include proofs of results announced in Bull. Acad. Polon. Sci. Cl. III **2** (1954), 371–373 and reviewed in MR **16**, 601.

H. M. Schaerf (St. Louis, Mo.)

367:

Burke, C. J.; and Rosenblatt, M. A Markovian function of a Markov chain. Ann. Math. Statist. **29** (1958), 1112–1122.

Let $X(n)$ $(n = 0, 1, 2, \cdots)$ be a Markov chain with a finite number of states having stationary transition matrix P. Let f be a function defined on the states. Then f determines a new process $f(X(n))$ $(n = 0, 1, \cdots)$. The authors determine conditions under which the new process is Markovian. Assume first that the chain has initial probability measures p chosen to make $X(n)$ stationary. Then $X(n)$ is reversible if $p_i p_{ij} = p_j p_{ji}$. Let S_α $(\alpha = 1, \cdots, r)$ be the sets of constancy for f.

Theorem: If $X(n)$ is reversible with $p_i > 0$ for all i, then $Y(n)$ is Markovian if and only if, for any fixed $\beta = 1, \cdots, r$, (1) $\sum_{i \in S_\beta} p_{ij} = C_{S_\alpha S_\beta}$ has the same value for all i in S_α $(\alpha = 1, \cdots, r)$. Condition (1) is not necessary for the general chain.

Theorem: A sufficient condition that $Y(n)$ be Markovian for every initial distribution of $X(n)$ is given by

$$\sum_{i \in S_\alpha} p_{ki} P_{i,S_\beta} = p_{k,S_\alpha} C_{S_\alpha,S_\beta}$$

for all k, α, β.

For continuous time chain $X(t)$ the analogue of (1) is almost necessary for $Y(t) = f(X(t))$ to be Markovian for

every initial distribution. Specifically the following theorem is proved. Theorem: Let $X(t)$ $(0 \leq t < \infty)$ be a Markov chain with a finite number of states and stationary transition matrix $P(t)$ such that $\lim_{t\downarrow 0} P(t) = I$. Then $Y(t) = f(X(t))$ is Markovian for every initial distribution if and only if for each $\beta = 1, \cdots, r$ separately either (i) $p_{i,S_\beta}(t) = 0$ for all $i \notin S_\beta$ or (ii) $p_{i,S_\gamma}(t) = C_{S_\beta,S_\gamma}(t)$ for every $i \in S_\beta$ and all $\alpha = 1, \cdots, r$.

Other related questions are discussed. The case of abstract space is considered.

J. L. Snell (Hanover, N.H.)

368:

Lévy, Paul. Processus markoviens et stationnaires. Cas dénombrable. Ann. Inst. H. Poincaré **16** (1958), 7–25.

The author indicates modifications and corrections required in a previous paper [Ann. Sci. École Norm. Sup. (3) **68** (1951), 327–381; MR **13**, 959]. New definitions of separability of a stochastic process and of related concepts are proposed.

J. L. Doob (Urbana, Ill.)

369:

Has'minskiĭ, R. Z. Diffusion processes and elliptic equations degenerating at the boundary of a region. Teor. Veroyatnost. i Primenen. **3** (1958), 430–451. (Russian. English summary)

Let L be the operator

$$\sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial u}{\partial x_i}$$

acting on functions on a region D of the plane. Here D lies in the half-plane $x_2 > 0$ and is bounded by an interval $\Gamma_0 = [r_1, r_2]$ on the x_1 axis, together with a smooth curve Γ joining the endpoints of Γ_0. The coefficients of L are supposed smooth in $D \cup \Gamma$, with L non-degenerate there, and Γ is supposed smooth enough so that all points of this curve are regular for the first boundary value problem for $Lu = 0$. No hypotheses are imposed on the coefficients of L as their arguments approach Γ_0 however, and the complications caused by this fact are the occasion of the paper. A boundary set γ is 'attracting' if, for every point x_0 of γ, and every $\eta > 0$, the probability that the limit points of a path from x in D lie within distance η of γ approaches 1 when $x \to x_0$. Here the 'paths' are the probability paths of the diffusion process determined by L. Conditions on the coefficients of L are found which are necessary and sufficient that Γ_0 be attracting, as well as that a point of Γ_0 be 'repulsive', a property tending in the opposite direction. (The discussion of these definitions is not entirely clear.) As already stated, the points of Γ are regular for the first boundary value problem. In probability language this means that the probability that a probability path from x has all its boundary limit points in a preassigned neighborhood of a point x_0 of Γ approaches 1 when $x \to x_0$. Conditions on the coefficients of L sufficient for regularity of a point of Γ_0 are found. Conditions sufficient that almost no probability path from some (equivalently every) point of D have a limit point on Γ_0 are found.

Examples involving further specialized hypotheses are worked out in some detail. Finally, Markov processes are considered in an open subset G of an abstract metric space, to obtain results connecting regularity of a boundary point in the above probabilistic sense with the property that probability paths from a point of G converge to single points of the boundary. This work is related to one by the reviewer [Proc. Third Berkeley Symposium Math. Statist. Probability, 1954–1955, vol. II, pp. 49–80, Univ. of California Press, Berkeley, 1956; MR **18**, 941].

J. L. Doob (Urbana, Ill.)

370:

Skorohod, A. V. Limit theorems for Markov processes. Teor. Veroyatnost. i Primenen. **3** (1958), 217–264. (Russian. English summary)

In a previous paper [Teor. Veroyatnost. i Primenen. **1** (1956), 289–319; MR **18**, 943], the author proved limit theorems for stochastic processes whose sample functions do not have oscillatory discontinuities. He now applies these theorems to Markov processes, incidentally correcting a statement in a previous note on the subject [Dokl. Akad. Nauk SSSR **106** (1956), 781–784; MR **17**, 1217]. Refining theorems of Kinney [Trans. Amer. Math. Soc. **74** (1953), 280–302; MR **14**, 772] and Dynkin [Izv. Akad. Nauk SSSR Ser. Mat. **16** (1952), 563–572; MR **14**, 567], the author finds a weaker condition on the transition probability distributions of a Markov process ensuring that almost all its sample functions have non-oscillatory discontinuities. The application of the first reference above to Markov processes is based on semigroup theory as applied to processes with stationary transition probabilities, and thereby to those with non-stationary transition probabilities, since these processes, considered on space-time have stationary transition probabilities. A typical theorem is the following, stated roughly here. Let $\{\xi_n(t), 0 \leq t \leq 1\}$ be a Markov process with stationary transition probabilities satisfying certain regularity conditions. It is supposed that, in a certain specified sense, the infinitesimal operator of the semigroup determined by the $\xi_n(t)$ process converges when $n \to \infty$ to that determined by the $\xi_0(t)$ process, and it is supposed that the distribution of $\xi_n(0)$ converges to that of $\xi_0(0)$. Then if F is a certain type of function of sample functions, the distribution of F on the $\xi_n(t)$ process converges to that of F on the $\xi_0(t)$ process.

J. L. Doob (Urbana, Ill.)

371:

Austin, D. G. Note on differentiating Markoff transition functions with stable terminal states. Duke Math. J. **25** (1958), 625–629.

Given Markov transition probabilities $p_{ij}(t)$: $i, j = 1, 2, \cdots, t \geq 0$, such that $p_{ij}(0+) = 0 (i \neq j) = 1 (i = j)$, Austin [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 224–226; MR **16**, 1130] proved that if $p_{ii}'(0) = \lim_{t\downarrow 0} t^{-1}[p_{ii}(t) - 1] > -\infty$ (i.e., if i is stable), then for each j, $p_{ij}'(t) \in C[0, +\infty)$ and $p_{ij}(t_1 + t_2) = \sum_k p_{ik}'(t_1) p_{kj}(t_2)$. He now proves that if j is stable, then for each i, $p_{ij}'(t) \in C[0, +\infty)$ and $p_{ij}'(t_1 + t_2) = \sum_k p_{ik}(t_1) p_{kj}'(t_2)$. He also notes (at the suggestion of K. L. Chung) that $p_{ij}'(+\infty) = 0$ if either i or j is stable.

H. P. McKean, Jr. (Cambridge, Mass.)

372:

Tortrat, Albert. Étude d'une méthode d'iteration propre à certaines matrices, application aux processus de Markhoff correspondants, cas des processus continus homogènes par rapport à l'espace, cas non homogène. Publ. Sci. Univ. Alger. Sér. A **4** (1957), 145–189.

The article is expository and discusses various boundary value problems for the one-dimensional random walk

with probability p of going one step to the right and $q=1-p$ of going one step to the left. The absorbing, reflecting, and partially reflecting barriers are discussed. Corresponding diffusion problems are treated by a passage to the limit. *M. Rosenblatt* (Bloomington, Ind.)

373:

Sjöberg, Boris. Über lineare Irrfahrt mit Absorptionsschranken. Acta Acad. Abo. **21** (1958), no. 13, 13 pp.

The author considers a random walk on a finite interval of length a. There is probability p of a unit step to the right, $1-p$ of one to the left, and the endpoints of the interval are absorbing barriers. He evaluates the probability that a particle will go from j to k in n steps and then, assuming that a particle is created at a specified point j at each time $0, 1, \cdots$ and then moves in the walk independently of other particles, he finds the probability that there will be m particles at time n in position k. When $n\to\infty$ there is a limiting distribution of particles. For example, if $p=\frac{1}{2}$, the expected number of unabsorbed particles is asymptotically $j(a-j)$, when the interval is $[0, a]$. *J. L. Doob* (Urbana, Ill.)

374:

Hammersley, J. M. Percolation processes: Lower bounds for the critical probability. Ann. Math. Statist. **28** (1957), 790–795.

The author continues here the study of percolation processes begun by S. R. Broadbent and J. M. Hammersley [Proc. Cambridge Philos. Soc. **53** (1957), 629–641; MR **19**, 989]; it is assumed that one is given a set of infinitely many atoms connected by some collection of directed bonds, but that there is only a probability $p<1$ that any given bond will be undammed (part of an allowed path). The problem considered here is to determine the largest value of p, p_d, for which there is probability 1 that for a given atom A the set of all undammed paths from A cover only finitely many atoms. Fewer assumptions are made here than in the authors' previous paper, only that (a) "the number of bonds from any atom is finite" and (b) "any finite subset of atoms contains an atom from which a bond leads to some atom not in the subset". Lower bounds on the critical probability p_d are found in terms of the expected number of new atoms that can be reached from a point A as the last steps in n-step walks along undammed paths. *G. Newell* (Stockholm)

375:

Basharin, G. P. An investigation, using probability theory, of a two-stage trunk-hunting telephone system with refused calls. Soviet Physics. Dokl. **121** (**3**) (1958), 713–717 (101–104 Dokl. Akad. Nauk SSSR).

A continuous-parameter process of Markov type in k dimensions is used as a model of a two-stage switching network in which calls are refused if they cannot be completed at once. The k random variables used, however, are a sufficiently detailed description of the state of the system to make the system Markovian only in two relatively trivial cases, in which all the blocking is due directly to the finite size of trunk groups or switches, and none occurs because of failure of idle links to match. In these cases the equilibrium equations can be solved to give formulas similar to Erlang's.

V. E. Beneš (Murray Hill, N.J.)

376:

Zítek, František. Zur Theorie der ordinären nachwirkungsfreien Folgen. Czechoslovak Math. J. **8** (**83**) (1958), 448–459. (Russian. German summary)

For finite streams of random events (non-stationary, in general) the author proves the following generalizations of a theorem of Korolyuk [A. Ya. Hinčin, Trudy Mat. Inst. Steklov. vol. 49, Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 276]. (1) A stream with absolutely continuous mean value function $M(t)$ is an ordinary one if and only if $\mu(t)=\lambda(t)$ a.e. Here $\mu(t)=M'(t)$ a.e.; $\lambda(t)=\lim_{\tau\to 0+}\psi_1(\tau, t)/\tau$, where $\psi_1(\tau, t)$ is the probability that least one event occurs in the time interval $(t, t+\tau)$. (2) A stream with continuous m.v.f. $M(t)$ is an ordinary one if and only if $M(I)=\Lambda(I)$ for every $I\subset\langle 0, T)$. Here $M(I)=M(\beta)-M(\alpha)$ for $I=(\alpha,\beta)$ and $\Lambda(I)$ is defined by Burkill's integral $\int_I \Lambda(K)$, where $\Lambda(K)=\psi_1(\delta-\gamma,\gamma)$ for $K=(\gamma,\delta)$.

Further, the author studies the k-ordinary streams and the quasi-stationary streams and gives a simple representation of Khintchin's χ_j-functions [Hinčin, Teor. Veroyatnost. i Primenen. **1** (1956), 3–18; MR **19**, 328] by a Burkill integral. *J. Janko* (Prague)

377:

Karlin, S.; Miller, R. G., Jr.; and Prabhu, N. U. Note on a moving single server problem. Ann. Math. Statist. **30** (1959), 243–246.

The moving server problem is that of B. McMillan and the reviewer [same Ann. **28** (1957), 471–478; MR **19**, 514]. It is identically the problem of the number served in a busy period of a single server queuing system (service in order of arrival) with Poisson arrivals and general distribution of service time, if the busy period is initiated by a demand with prescribed service time, the recognition of the last clause being the contribution of the present authors. With this identification and some results of L. Takacs [Acta Math. Acad. Sci. Hungar. **6** (1955), 101–129; MR **17**, 51] the authors are able to give not only a concise proof of the integral equation conjectured by McMillan and Riordan, but also a similar equation for the Laplace-Stieltjes transform of the distribution of the length of such a busy period. These equations are solved for the two special cases of constant and exponential service time distributions. *J. Riordan* (New York, N.Y.)

378:

Beneš, Václav E. On trunks with negative exponential holding times serving a renewal process. Bell System Tech. J. **38** (1959), 211–258.

Incoming calls are served by a group of N trunks. Calls arriving when all trunks are occupied are lost. The duration of a call has an exponential distribution with mean $1/\gamma$. The intervals between incoming calls are independent, with an arbitrary distribution A independent of time. The process is studied by means of the imbedded Markov chain $N(1), N(2), \cdots$, where $N(k)$ is the number of trunks found busy by the kth arriving customer. The generating function

$$\psi_n(z) = \sum_{k\geq 0} z^k \Pr\{N(k)=n\}$$

is found. If $N(0)=0$ this reduces to a result of Pollaczek [C. R. Acad. Sci. Paris **236** (1953), 1469–1470; MR **14**, 773]. The limiting probabilities $p_n=\lim_{k\to\infty}\Pr[N(k)=n]$, known previously, are derived from this. Bounds and approximations for p_N, the probability of a loss, are

obtained. Considering the loss p_N as a functional of the distribution A of inter-arrival times, it is shown that for fixed γ and fixed mean μ_1 of A, the loss is minimized by taking A as the step function concentrating mass 1 at μ_1. On the other hand, keeping γ and μ_1 fixed, the loss p_N can be made arbitrarily close to 1 by using a distribution A having all its weight on two steps. If p_N and γ are fixed, the distribution A maximizing the amount of traffic handled (it has all its weight on one step) is found. The mean and covariance of $\{N(k)\}$ are studied. There are numerous charts. *T. E. Harris* (Santa Monica, Calif.)

379:

Ionescu Tulcea, C. On a class of operators occurring in the theory of chains of infinite order. Canad. J. Math. **11** (1959), 112–121.

Let T and E be measurable spaces. If x is in E let u_x be a mapping of T into T. If t is in T and if A is a measurable subset of E, suppose that $p(t, A)$ defines a probability measure for fixed t, and define Uf, for f bounded and measurable on T, by

$$(Uf)(t) = \int_E p(t, dx)f[u_x(t)].$$

Under supplementary hypotheses too detailed to repeat here, ergodic properties of the sequence of powers of U are proved which contain various previous results proved in studies of chains of infinite order by Onicescu and Mihoc [Bull. Sci. Math. **59** (1935), 174–192] and later writers. Under suitable restrictions, with T the class of non-positive integers, it is shown that p is the transition function of a stationary mixing stochastic process, for which a form of the central limit theorem is stated.

J. L. Doob (Urbana, Ill.)

380:

Watanabe, Hisao; and Motoo, Minoru. Ergodic property of recurrent diffusion processes. J. Math. Soc. Japan **10** (1958), 272–286.

Let $X(t, \omega)$ be a diffusion process with state space $[a, b]$, $-\infty \leq a < b \leq \infty$. Let $X^{(x)}(t, \omega)$ be a sample function starting from x at time 0, $a < x < b$. The process is called "recurrent" if $P_x\{X^{(x)}(t) = y$ for some $t\} = 1$ for each $x, y \in (a, b)$. Associated with such a process is a measure m, which is the measure v of Feller [Ann. of Math. (2) **61** (1955), 90–105; MR **16**, 824]; e.g., for the Wiener process on the infinite line, m is Lebesgue measure. In general, m will attach positive weights to the points a and b. Let f and g be m-integrable functions such that $\int_a^b g(x)dm(x) \neq 0$. The main result is that with probability 1,

$$\lim_{T\to\infty} \int_0^T f[X^{(x)}(t, \omega)]dt \Big/ \int_0^T g[X^{(x)}(t, \omega)]dt = \int f dm / \int g dm.$$

Related results were given by the reviewer and H. E. Robbins for discrete-parameter Markov processes [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 860–864; MR **15**, 140]. The author's methods are related to those of Chung [Trans. Amer. Math. Soc. **76** (1954), 396–419; MR **16**, 149] and Derman [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1155–1158; MR **16**, 495]. The method involves picking a pair of points x, y and defining t_1 as the first time $X^{(x)}(t)$ reaches y, t_2 as the first subsequent time it reaches x, t_3, as the first time afterwards that it reaches y, etc. The strong law of large numbers is then applied to the sequence of independent random variables $Y_i = \int_{t_{2i-2}}^{t_{2i}} f[X^{(x)}(t)]dt$, the expectation of Y_i being evaluated by a series of preparatory lemmas; similarly for g.

T. E. Harris (Santa Monica, Calif.)

381:

Hyvärinen, L. P. The autocorrelation and the power spectrum of nonstationary shot noise. Acta Polytech. Scandinav. **252** (1958), 23 pp.

By non-stationary shot noise the author means shot noise for which the probability of one shot in the interval $(t, t+dt)$ is $h(t)dt$, while the probability of more than one shot in $(t, t+dt)$ is of a lower order of magnitude. (The usual case of stationary shot noise corresponds to setting $h(t) = \text{const.}$) The probability $P(N)$ of N shots occurring in the interval (t_1, t_2) is found to be given by a natural generalization of the Poisson distribution, i.e.

$$(1) \qquad P(N) = \frac{(\bar{n}\tau)^N}{N!} \exp(-\bar{n}\tau),$$

where $\tau = t_2 - t_1$ and $\bar{n} = \tau^{-1} \int_{t_1}^{t_2} h(t)dt$. The author studies the non-stationary processes

$$i_\delta(t) = \sum_k \sigma(t-t_k) \quad \text{and} \quad i(t) = \sum_k i_0(t-t_k),$$

where the t_k are occurrence times distributed according to (1) and δ is the Dirac delta function. For the autocorrelation functions of these processes he finds

$$R_\delta(\tau) = \bar{n}\delta(\tau) + R_h(\tau),$$
$$R_i(\tau) = i_0(\tau) * i_0(-\tau) * R_\delta(\tau),$$

respectively, where

$$R_h(\tau) = \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} h(t)h(t-\tau)dt$$

and $*$ denotes the operation of convolution. The corresponding power spectra are derived, and the case where $h(t)$ is a periodic function is examined.

R. A. Silverman (New York, N.Y.)

382:

Friedland, Bernard. Least squares filtering and prediction of non-stationary sampled data. Information and Control **1** (1958), 297–313.

This is an extension of the work of Bode and Shannon [Proc. I.R.E. **38** (1950), 417–425; MR **11**, 672], Booton [ibid. **40** (1952), 977–981] and others to discrete time parameter non-stationary processes. By using the ensemble shaping technique, the determination of an optimum filter is reduced essentially to factoring the covariance matrix Φ into a product of a lower semimatrix C and its transpose C'. It is shown how the elements of C may be obtained recursively, with the necessary computations being well within the capabilities of small scale digital computers. Explicit expressions for the optimal filter characteristics are derived for several special cases. *L. A. Zadeh* (New York, N.Y.)

383:

Moran, P. A. P. A probability theory of a dam with a continuous release. Quart. J. Math. Oxford Ser. (2) **7** (1956), 130–137.

In the present note the author studies, in the framework of his probability theory of dams, the case of the discrete distributions of the random variables X_t (the amount of

the flow into the dam at the beginning of the year) and Z_t (the amount of water in the dam before the input). The rule of release chosen by the author is the following: The amount of water released after the input is X_t+Z_t or M, whichever is less, where M is a characteristic of the dam, the capacity of the dam being K.

If p_i $(i=0, 1, \cdots)$ is the probability distribution of X_t and P_i $(i=0, 1, \cdots)$ that of Z_t, then they are related by an infinite system of equations, which becomes in the case $K=\infty$,

$$\begin{aligned} P_0 &= P_0(p_0+\cdots+p_M)+\cdots+P_Mp_0,\\ P_1 &= P_0p_{M+1} \qquad\qquad +\cdots+P_Mp_1+P_{M+1}p_0,\\ &\cdot \qquad \cdot \qquad\qquad\qquad +\cdots+ \quad \cdot \qquad \cdot \end{aligned}$$

representing the equation of a Markoff chain, to be found as well in the theory of queues of Kendall (quoted by the author).

Assuming $M=1$ and using for the two distributions the generating functions $p(z)=\sum p_iz^i$ and $P(z)=\sum P_iz^i$ we obtain the relation $P(z)=p_0P_0(z-1)/(1-p(z))$. Using this theory of discrete distributions the author goes over to continuous distributions, considering, especially the case when the density of the input during the time t is of the form $\lambda x^{t-1}e^{-\mu x}$. *O. Onicescu* (Bucharest)

384:

Downton, F. A note on Moran's theory of dams. Quart. J. Math. Oxford Ser. (2) 8 (1957), 282–286.

F. Downton uses the analogy between the equations of the Markoff process, which are met in Moran's theory of dams and in the theory of queues, and generalizes Moran's transition process from the discrete distribution of the random variables associated in the process to the continuous distribution.

The author obtains with these methods more general results than those known previously.

O. Onicescu (Bucharest)

385:

Baticle, Edgar. Le mélange des cartes par coupes successives. C. R. Acad. Sci. Paris **248** (1959), 1284–1286.

STATISTICS

See also 36, 362, 382.

386:

★Fraser, D. A. S. Statistics: an introduction. A Wiley Publication in Mathematical Statistics. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. ix + 398 pp. $6.75.

This book was prepared as a text for a two-semester course in mathematical statistics for those with the background of a full year of calculus. The purpose of the book is to develop in a logical manner the framework of mathematical statistics without long diversions into applications. The chapter topics are: 1, Introduction; 2, probability, introduced via an empirical frequency limit and applied to elementary combinatorial situations; 3, discrete distributions; 4, continuous distributions; 5, means, variances, generating functions, etc.; 6, sample moments, laws of large numbers, central limit theorem; 7, sampling from a finite population, along the lines principally developed by Tukey, dealing with single classification and two-way classifications, both nested and crossed; 8, linear transformations, the χ^2- and F-distributions; 9, point estimation, including sufficiency, Blackwell's theorem and maximum likelihood; 10, hypothesis testing, including the Neyman-Pearson lemma and the likelihood ratio method; 11, confidence intervals; 12, regression analysis, with a good account of fitting constants in stages; 13, factorial designs with up to three factors. Three short chapters on experimental design, sampling inspection and non-parametric methods, together with tables of the normal, t-, F- and χ^2-distributions complete the book.

A most pleasing feature of the book is the modernity of the outlook. This is particularly noticeable in the whole of chapter 7, in the use of Blackwell's theorem and in the emphasis on linear transformations and least squares. Probably no other book deals with these topics as well as the present one, and they surely have a strong case for inclusion in a modern text-book. There are large numbers of good exercises, many of them of the standard normally expected of an English text-book but rarely found in an American one. (The author is in Canada.) No gross errors were detected and minor ones are rare. (The fundamental theorem of the integral calculus is wrongly stated on p. 61.)

A weak feature is the style. Many sections are quite difficult and could have been improved by a more careful writing. For example, the notions of conditional probability and expectations are cursorily treated and this leads to obscurities in later sections: the definition of unbiasedness would have read better as $E(x|\theta)=\theta$ rather than as $E(x)=\theta$ (p. 215); confidence limits would have been easier to understand had equation (1.4) on p. 277 had $P_r\{\cdot|\mu\}$ instead of an absolute probability. The idea of independence of several events is never explained. The introductions of inference ideas occur amidst some complicated algebra (p. 140) and in the middle of the central limit theorem (p. 123). A careful discussion of hypothesis testing and confidence intervals in elementary cases, before dealing with general theory or general linear situations, would have improved the book. Occasionally the ideas are more complicated than they need be. For example, a proof of the Halmos-Savage theorem at the level of this book is easy (p. 219)—the student will wonder what the trouble can be—and it is surely unnecessary in explaining power (p. 248) to do it for the t-test rather than the case where σ is known. It is surprising that fiducial ideas should be mentioned, yet there is no mention at all of Bayes' theorem, even in a frequency sense.

These weaknesses of style will probably worry a student working on his own, but any teacher giving a course at this level should seriously consider using this modern, accurate text. The choice of topics is good and deficiencies of exposition are not so serious that they cannot be easily surmounted by the teacher.

D. V. Lindley (Cambridge, England)

387:

Fisher, Walter D. On grouping for maximum homogeneity. J. Amer. Statist. Assoc. **53** (1958), 789–798.

The objective of this paper is the determination of a practical procedure for grouping a set of arbitrary numbers so that the within group variance is maximized. The treatment of the paper is limited to discrete distributions

though references are given to similar problems involving continuous distributions. The problem is the grouping of K elements into G groups, $G<K$, so that the squared distance $D=\sum_{i=1}^{K} w_i (a_i-\bar{a}_i)^2$ is minimized. Here w_i is the known weight associated with the element, a_i is the numerical measure of the element, and $\bar{a}_i$ is the weighted mean of those measures in the same group. A system of grouping is called a partition.

The author distinguishes (1) the unrestricted problem and (2) the restricted problem in which a priori conditions are imposed. In each case the solution is combinatorial. The chief contribution of the paper is in establishing principles or lemmas which restrict the number of potential partitions to a relatively small subset. The paper contains a description of an automatic computer program capable of handling problems with K up to 200 and G up to 10. *P. S. Dwyer* (Ann Arbor, Mich.)

388:

Levine, Jack. Monomial-monomial symmetric function tables. Biometrika **46** (1959), no. 1/2, 205–213.

389:

Hájek, Jaroslav. On the distribution of some statistics in the presence of intraclass correlation. Časopis Pěst. Mat. **83** (1958), 327–329. (Czech. Russian and English summaries)

A theorem for the n-dimensional normal distribution is proved which states that the results concerning independent variates are also applicable—after a slight modification—in the case when intraclass correlation is present. The theorem is then applied to some important statistics; it is proved, for instance, that the distributions of maximum-likelihood estimates for regression coefficients and related statistics under the assumption of the presence of intraclass correlation are the same as in the case of independent variates with a modified variance.

J. Janko (Prague)

390:

Newman, D. J.; and Klamkin, M. S. Expectations for sums of powers. Amer. Math. Monthly **66** (1959), 50–51.

A sequence of independent random variables with uniform density over the interval (0, 1) is sampled until the sum of their nth powers exceeds 1. The expected number $\mathscr{E}_n$ of trials is calculated and shown to be asymptotically equal to cn, c being given.

S. W. Nash (Vancouver, B.C.)

391:

Sankaran, Munuswamy. On the non-central chi-square distribution. Biometrika **46** (1959), no. 1/2, 235–237.

392:

Schmid, Paul. On the Kolmogorov and Smirnov limit theorems for discontinuous distribution functions. Ann. Math. Statist. **29** (1958), 1011–1027.

Let $X_1, \cdots, X_n$ be a sample of the random variable X with distribution function $F(x)$, and let $S_n(x)$ be the corresponding empirical distribution function. For $F(x)$ continuous, asymptotic distributions of the statistics

$$\sup_{-\infty<x<\infty} |S_n(x)-F(x)|, \quad \sup_{-\infty<x<\infty} \{S_n(x)-F(x)\},$$

$$\sup_{a\le F(x)} |[S_n(x)-F(x)]/F(x)|, \quad \sup_{a\le F(x)} \{[S_n(x)-F(x)]/F(x)\}$$

are known and are independent of $F(x)$. In the present paper asymptotic distributions are obtained for these four statistics under the assumption that $F(x)$ has discontinuities. These asymptotic distributions depend only on the values $F(x_i-0)$ and $F(x_i)$, where the x_i are the discontinuity points of $F(x)$.

Z. W. Birnbaum (Seattle, Wash.)

393:

Trybula, S. On the minimax estimation of the parameters in a multinomial distribution. Zastos. Mat. **3** (1958), 307–322. (Polish. Russian and English summaries)

A minimax estimator is found, and proved to be unique, for the parameters of a multinomial distribution when the loss function is a linear combination (with non-negative coefficients) of the squared errors. Although this is a generalization of the result obtained, for a binomial distribution, by J. L. Hodges Jr. and E. L. Lehmann [Ann. Math. Statist. **21** (1950), 182–197; MR **12**, 36], the approach is rather different from theirs. Following H. Steinhaus [ibid. **28** (1957), 633–648; MR **19**, 1095], the problem is treated as one of a game between the statistician and the devil (nature), whose optimum strategy is given as well. The similar problem when the random variable takes a denumerable number of values is also solved. *S. K. Zaremba* (Swansea)

394:

Tiago de Oliveira, J. Estimators and tests for continuous populations with location and dispersion parameters. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 121–146.

Suppose n independent random variables have a common distribution law, known as to form, but not as to location and scale parameters. The author describes methods of testing and estimating these unknown parameters, based on the ordered values of the random variables.

M. Dwass (Evanston, Ill.)

395:

Lloyd, D. E. Note on a problem of estimation. Biometrika **46** (1959), no. 1/2, 231–235.

396:

Leslie, D. C. M. Determination of parameters in the Johnson system of probability distributions. Biometrika **46** (1959), no. 1/2, 229–231.

397:

Ramachandran, K. V. On the studentized smallest chi-square. J. Amer. Statist. Assoc. **53** (1958), 868–872.

Let k F statistics $F_i=mS_i/tS$ be defined by means of mutually independent statistics $S_1, S_2, \cdots, S_k$ and S, where S_i/σ^2 $(i=1, 2, \cdots, k)$ and S/σ^2 have chi-square distributions with t and m degrees of freedom respectively. This paper gives the lower 5% points of the statistics $v=mS \min/tS$ for different values of t, m and k, as a part of percentage points of the studentized largest chi-square to be published soon. Various applications of the studentized smallest chi-square are mentioned briefly.

T. Kitagawa (Fukuoka)

398:

Tumanyan, S. H. On the power of the chi-square criterion applied to the problem of two samples relative to "near" alternatives. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 6, 31–45. (Russian. Armenian summary)

The author considers two independent samples of m and n observations respectively drawn from two populations with distribution functions $F(x)$ and $G(x)$. The observations are classified into $s+1$ mutually exclusive and exhaustive groups. He constructs the statistic

$$(1) \qquad x^2 = mn \sum_{k=0}^{s} \frac{1}{\mu_k+\nu_k}\left(\frac{\mu_k}{m}-\frac{\nu_k}{n}\right)^2,$$

where μ_k and ν_k are the number of observations belonging to the kth group ($k=0, 1, \cdots, s$) corresponding to the first and second samples.

It is known that if the hypothesis H_0 ($F\equiv G$) is true, the statistic (1) has in the limit, as $m\to\infty$ and $n\to\infty$, a chi-square distribution with s degrees of freedom. The author studies the case when H_0 is not true. Let p_k be the probability that an observation of the first sample will belong to the kth group and p_k', the corresponding probability for the second sample. He assumes that

$$(2) \qquad p_k'-p_k = \frac{z_k((m+n)p_k)^{\frac{1}{2}}}{(mn)^{\frac{1}{2}}},$$

where z_k is a constant, $k=0, 1, \cdots, s$. He shows that under the condition (2) the statistic (1) is, in the limit as $m\to\infty$ and $n\to\infty$, distributed as a non-central chi-square with s $d\cdot f$ and non-centrality parameter $\sum_{k=0}^{s} z_k^2$.

R. G. Laha (Washington, D.C.)

399:

Dunn, Olive Jean. Estimation of the means of dependent variables. Ann. Math. Statist. **29** (1958), 1095–1111.

Let y_i with $i=1, \cdots, k$ be jointly distributed variables with means μ_i. A set of simultaneous confidence intervals for the means with confidence coefficient $1-\alpha$ consists of $2k$ sample functions g_i and h_i with the property that, if E_i is the event that g_i to h_i covers μ_i, then the probability that the E_i occur simultaneously $\geqq 1-\alpha$. Methods are given for constructing sets of simultaneous confidence intervals for the means of variables which follow a multivariate normal distribution. The condition of normality may sometimes be relaxed for large samples.

Exact confidence intervals using independent linear combinations of the sample values are first established (a) for the case of known variances and (b) for the case of unknown but equal variances. The later sections feature different methods for obtaining confidence intervals of bounded confidence level. Fixed lengths are obtained when the variances are known and variable lengths result when the variances are unknown but assumed equal. Treatment includes the case of linear contrasts, the use of the Bonferroni inequality, and inequalities between the independent and the dependent cases. Results are summarized in Table I and are tabulated for different values of k in Table II.

P. S. Dwyer (Ann Arbor, Mich.)

400:

Gnedenko, B. V. On the Wilcoxon test of comparing of two samples. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 611–614. (Russian. English summary)

An example is given to show that the Wilcoxon two sample test [F. Wilcoxon, Biometrics Bull. **1** (1945), 80–83] is inconsistent for certain alternatives. [On p. 618, read $DU=\frac{1}{4}m^2n$ for $DU=\frac{1}{4}m\,n$.]

W. Hoeffding (Chapel Hill, N.C.)

401:

Ghosh, S. P. A note on stratified random sampling with multiple characters. Calcutta Statist. Assoc. Bull. **8** (1958), 81–90.

Minimization of the generalized variance [H. Cramér, *Mathematical methods of statistics*, Princeton Univ. Press, 1946; MR **8**, 39; sec. 22.7) is proposed as a criterion for optimum allocation of observations to strata when several characters are of interest. A numerical error results in erroneous criticism of an earlier criterion [W. G. Cochran, *Sampling techniques*, Wiley, New York, 1953; MR **14**, 887; sec. 5.11].

P. Meier (Baltimore, Md.)

402:

Somers, Robert H. The rank analogue of product-moment partial correlation and regression, with application to manifold, ordered contingency tables. Biometrika **46** (1959), no. 1/2, 241–246.

This is a formal-descriptive, non-probability approach.

I. R. Savage (Minneapolis, Minn.)

403:

Srivastava, A. B. L. Effect of non-normality on the power of the analysis of variance test. Biometrika **46** (1959), no. 1/2, 114–122.

Let $x_{ij}=A+B_j+\varepsilon_{ij}$, $i=1, \cdots, n$; $j=1, \cdots, k$, where A and B_j are constants with $\sum B_j=0$, and the ε_{ij} are random variables with mean zero and variance σ^2. The author studies the power of the one-way analysis of variance test in the case when the ε_{ij}/σ, instead of being normally distributed, have a density of the form

$$f(x) = \phi(z)-\frac{\lambda_3}{6}\phi^{(3)}(x)+\frac{\lambda_4}{24}\phi^{(4)}(x)+\frac{\lambda^2{}_3}{72}\phi^{(6)}(x),$$

where $\phi^{(k)}(x)$ is the kth derivative of the standardized normal density function and λ_3 and λ_4 are the standardized third and fourth cumulants of the distribution of the ε_{ij}. This is accomplished by deriving the non-central distribution of the F-ratio. It is found that a small deviation from normality (λ_3 and λ_4 small) has no great influence on the power. Several tables giving numerical examples are included.

J. R. Blum (Bloomington, Ind.)

404:

Mitton, R. G.; and Morgan, F. R. The design of factorial experiments: a survey of some schemes requiring not more than 256 treatment combinations. Biometrika **46** (1959), no. 1/2, 251–259.

405:

Elfving, G. A selection problem in experimental design. Soc. Sci. Fenn. Comment. Phys.-Math. **20** (1957), no. 2, 10 pp.

Let $\alpha_1, \cdots, \alpha_m$ be unknown parameters and let $\alpha=(\alpha_1, \cdots, \alpha_m)'$ be the parameter vector. Let x_i, $i=1, \cdots, N$ be uncorrelated random variables with mean values $u_i'\alpha$ and unit variances, where u_i', $i=1, \cdots, N$ are known vectors. For a given known vector of coefficients $c=(c_1, \cdots, c_m)'$, the problem considered here is to select a set of n of the random variables $x_1, \cdots, x_N$ which will

minimize the variance of the least-squares estimator $\hat{\theta}$ of $\theta = c'\alpha$ utilizing only n random variables selected from $x_1, \cdots, x_N$. The variance of $\hat{\theta}$ for any particular selection w of n of these random variables, is given by $c'M^{-1}c$, where M is the $m \times m$ matrix $\sum_w u_i u_i'$, $\sum_w$ denoting summation over the n particular values of i in selection w. The author approximates the solution to this selection problem in a sense defined below. He shows that if M_* is a matrix defined by

$$M_* = \sum_{i=1}^{N} p_i u_i u_i',$$

where $0 \leq p_i \leq 1$, $\sum_{i=1}^{N} p_i = n$, then $V_* = c'M^{-1}c$ has a minimum for variations of the p's and that the minimum occurs at a point in the space of the p's for which the number of fractional p's is at most m and the number of non-zero p's is at most $n+m-1$. Thus, by selecting all of the random variables $(x_1, \cdots, x_N)$ corresponding to non-zero p's—an over selection by at most $m-1$ variables—the variance of $\hat{\theta}$ for such a selection will not exceed that for the optimum choice of only n of the random variables.

S. S. Wilks (Princeton, N.J.)

406:

★Driml, Miloslav; and Špaček, Antonín. Continuous random decision processes controlled by experience. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 43–60. Publishing House of the Czechoslovak Academy of Sciences. Prague, 1957. 354 pp. Kčs. 34.00.

A 'decision process controlled by experience' is formalized as follows. Let A be a non-empty set and let ω vary in a probability measure space Ω. Let t be a point of a given direction, and suppose that $x_t(\omega, a)$, $y_t(\omega, a)$ (where a is in A) are mappings into metric spaces X and Y respectively. It is supposed that $\lim_t x_t(\omega, a) = x(a)$ and $\lim_t y_t(\omega, a) = y(a)$ with probability 1. Under certain conditions, some of which are described in this paper, if x_0 is a specified element of X, there is a mapping $a_t(\omega)$ into A, depending on $y_t(\omega, a)$, for which $\lim_t x_t[\omega, a_t(\omega)] = x_0$.

J. L. Doob (Urbana, Ill.)

407:

Kovalenko, I. N. On one class of optimal resolving functions for a binomial family of distributions. Teor. Veroyatnost. i Primenen. 4 (1959), 101–105. (Russian. English summary)

Given a sequence of independent random variables $\eta_1, \eta_2, \cdots$, where $\eta_n = 1(0)$ with probability $p(1-p)$, and two possible decisions d_1, d_2, the problem is to find a decision function which minimizes the expected sample size at $p = p_0$ subject to upper bounds on the probabilities of making decision $d_2(d_1)$ when $p = p_1(p_2)$, where $p_1 < p_0 < p_2$. The proposed solution uses a Bayes procedure with constant cost and the "working backward" method. {For a simpler approach where, in effect, the cost per observation depends on p and is zero for $p \neq p_0$ compare Kiefer and Weiss, Ann. Math. Statist. 28 (1957), 57–74 [MR 19, 333].}

W. Hoeffding (Chapel Hill, N.C.)

408:

Bhate, D. H. Approximation to the distribution of sample size for sequential tests. I. Tests of simple hypothesis. Biometrika 46 (1959), no. 1/2, 130–138.

Let $f(x;\theta)$ be the probability density of a random variable X depending on a parameter θ. Let $X_j = (X_1, X_2, \cdots, X_j)$ be a sequence of independent observations of X. A sequential procedure for testing $H_0: \theta = \theta_0$ against the alternative $H_1: \theta = \theta_1$ may be constructed by dividing the sample space into three mutually exclusive regions A_j, R_j, C_j such that if: (1) $X_i \in C_i$ $(i = 1, 2, \cdots, j-1)$, $X_j \in A_j$, H_0 is accepted and observations terminate; (2) $X_i \in C_i$ $(i = 1, 2, \cdots, j-1)$, $X_j \in R_j$, H_0 is rejected and observations terminate; (3) $X_i \in C_i$ $(i = 1, 2, \cdots, j)$, a further observation is taken.

Also let N be the random variable defined as the smallest integer such that $X_N \notin C_N$, i.e., N is the sample size or decisive sample number. Asymptotic approximations to the distribution of N have been given by Wald [Ann. Math. Statist. 16 (1945), 117–156; MR 7, 131], by Kac [ibid. 16 (1945), 62–67; MR 6, 233] and by Darling and Siegert [ibid. 24 (1953), 624–639; MR 15, 449].

In this paper simple considerations of possible paths of the sequential sample lead to formulae for bounds on the distribution of N expressed in general terms. Some simplification is made by choosing

$$\alpha = \Pr[\text{rejecting } H_0 | H_0 \text{ true}] =$$
$$\beta = \Pr[\text{rejecting } H_1 | H_1 \text{ true}].$$

The formulae are evaluated for some special cases (e.g. the sequential probability ratio test for the mean of a normal distribution with known variance) and compared with some empirical data.

D. G. Chapman (Seattle, Wash.)

409:

Adke, S. R.; and Dharmadhikari, S. W. Gain due to sequential sampling from Gamma population. Calcutta Statist. Assoc. Bull. 8 (1958), 91–94.

Consider a random variable X having a Gamma type distribution described by the p.d.f.

$$(*) \qquad f(x;\sigma,l) = \begin{cases} (x^{l-1}/\sigma^l\Gamma(l))\exp(-x/\sigma), & x>0, \\ 0, \text{ elsewhere.} \end{cases}$$

Assuming that the parameter l is fixed, the authors test the simple hypothesis $H_0: \sigma = \sigma_0$ against the simple alternative $H_1: \sigma = \sigma_1$ with prescribed Type I error α and prescribed Type II error β. Let $n(\alpha, \beta)$ be the fixed sample size required by the Neyman-Pearson test and let $E_\sigma(n)$ be the expected number of observations required to reach a decision using the Wald sequential probability ratio test, when σ is the true value of the unknown parameter in (*). It is well-known from general considerations that both $E_{\sigma_0}(n)/n(\alpha, \beta)$ and $E_{\sigma_1}(n)/n(\alpha, \beta)$ are substantially less than one. The main point of the paper is that the ratios are independent of l. Thus, in order to evaluate the ratios numerically, one can use results obtained for the exponential distribution $(l=1)$.

Benjamin Epstein (Stanford, Calif.)

410:

★Duncan, Acheson J. Quality control and industrial statistics. Revised ed. Richard D. Irwin, Inc., Homewood, Ill., 1959. xxxiii+946 pp. $9.00.

Quality control, in essence, is the study of methods to maximize production, efficiency, profits, or some combination of these in the manufacturing process. As such it includes many management aspects. The present book explores the statistical aspects of the subject. A revision of a successful text, it has been expanded over one-third to include new topics of multiple fraction defective

sampling plans, acceptance sampling by variables, new government standards, rectifying inspection, the design of experiments, response surface methodology, and a much fuller treatment of the analysis of variance. These topics are in addition to the usual ones of acceptance sampling, control charts, regression theory, and the testing of hypotheses. The treatment is up-to-date, scholarly, and practical. Commendable are the many examples from industry, the consistent use of O.C. curves, the profusion of graphs, the full references, a brief account on the economic design of a quality control chart, the systematic use of the range in place of the standard deviation, and the estimation of the precision of measuring instruments and product variability. Less commendable are the sketchy inaccurate statements concerning the distribution of the product and quotient of two normally distributed independent variables, on pages 73–74. The Aspin-Welch test for the confidence limits of the difference of two means when $\sigma_1 \neq \sigma_2$ is given. The reviewer feels that the exact test of Scheffe [Ann. Math. Statist. **14** (1943), 35–44; MR **4**, 221] is preferable. Box [Biometrika **40** (1953), 318–335; MR **15**, 453] has demonstrated that the Bartlett test for homogeneity of a set of variances is really a test for kurtosis, and hence this test should not be used.

L. A. Aroian (Culver City, Calif.)

411:

Ekman, Gunnar. An approximation useful in univariate stratification. Ann. Math. Statist. **30** (1959), 219–229.

A population distributed over (a, b) with smooth density f is to be so divided into n strata, by points $a < x_1 < \cdots < x_{n-1} < b$, that stratified sampling will be most efficient. The author suggests that an approximate solution is given by making $(x_h - x_{h-1})P_h$ constant, where P_h is the area under f between x_{h-1} and x_h. This is simpler than, and usually nearly the same as, the rule of Dalenius and Gurney, who make $\sigma_h P_h$ constant, since in a narrow stratum the standard deviation σ_h is about $(x_h - x_{h-1})/\sqrt{12}$. Extensions, numerical considerations, and examples.

J. L. Hodges, Jr. (Berkeley, Calif.)

412:

Goodman, Leo A. On some statistical tests for mth order Markov chains. Ann. Math. Statist. **30** (1959), 154–164.

Given an observed sequence $\{X_1, X_2, \cdots, X_n\}$ from a stochastic process, there are in the literature two types of statistic, ψ^2 (more or less quadratic), and likelihood-ratio (more or less logarithmic), for testing a hypothesis of Markovity of one order, either fully specified or not, within a hypothesis of higher order. This paper is a detailed study of the asymptotic behaviour and relationships between these two types of statistic and of new generalizations of them. Previous papers by other authors are corrected and some of the reviewer's conjectures proved or disproved. There are references to papers on Markov chains by T. W. Anderson, M. S. Bartlett, P. Billingsley, R. Dawson, C. Derman, I. J. Good, L. A. Goodman, P. G. Hoel, V. E. Stepanov, and P. Whittle.

I. J. Good (Teddington)

413:

Parzen, Emanuel. On asymptotically efficient consistent estimates of the spectral density function of a stationary time series. J. Roy. Statist. Soc. Ser. B. **20** (1958), 303–322.

The article for the most part is a discussion of results obtained by the author [see Stanford University Appl. Math. Statist. Lab. Tech. Rep. no. 36]. A variety of spectral estimates in the case of stationary time series are considered. Under rather detailed assumptions on the asymptotic behavior of the covariance function $R(v)$ of the stochastic process such as $|R(v)| \leq R_0 e^{-\rho|v|}$, $\rho > 0$, or $\lim_{v\to\infty} v^r |R(v)| = R_r$, $r > 0$, estimates are produced which have asymptotically best behavior in the sense of some measure like mean square error. The character of the "optimal" estimates depends on the rate of decay of the covariance function. One might feel that such an assumption on the rate of decay would not be verifiable unless there was a refined theory in the context at hand specifying the rate of decay.

M. Rosenblatt (Bloomington, Ind.)

414:

Walker, A. M. The existence of Bartlett-Rajalakshman goodness of fit G-tests for multivariate autoregressive processes with finitely dependent residuals. Proc. Cambridge Philos. Soc. **54** (1958), 225–232.

The title tests [J. Roy. Statist. Soc. **15** (1953), 107–124; MR **15**, 333] require the existence of a "G-operator". Assuming a certain determinant of co-variances with lags to be nonzero, the author obtains an explicit formula for the operator. A counterexample shows that the G-operator in general does not exist.

H. Wold (New York, N.Y.)

415:

Hannan, E. J. The estimation of the spectral density after trend removal. J. Roy. Statist. Soc. Ser. B. **20** (1958), 323–333.

Given a process

$$y(t) = \sum \beta_j x_j(t) + \varepsilon(t),$$

where the residual process $\{\varepsilon\}$ is assumed to be stationary and to possess a spectral density $f(\theta)$. The author wishes to calculate the bias introduced in the estimate of $f(\theta)$ if the β_j are estimated by raw least-squares and ε estimated from $\hat{\varepsilon} = y - \sum \hat{\beta}_j x_j$. He finds that when the sequences $x_j(t)$ obey the conditions proposed by Grenander [Ann. Math. Statist. **25** (1954), 252–272; MR **15**, 973], which ensure the efficiency of the estimates $\hat{\beta}_j$, then the bias introduced in the estimate of $f(\theta)$ is, up to order n^{-1}, a simple multiple of $f(\theta)$. From this result he derives expressions for the bias in the estimated autocorrelation coefficients.

P. Whittle (Wellington)

416:

Zubrzycki, S. Remarks on random, stratified and systematic sampling in a plane. Colloq. Math. **6** (1958), 251–264.

Consider a region D in the plane and a stochastic process $Y(p)$, $p \in D$, with the properties: (a) the probability distribution of $Y(p)$ is the same for every $p \in D$; (b) the correlation function $R[Y(p), Y(q)] = f(pq)$ depends only on the vector pq for all $p, q \in D$; (c) $\lim_{|pq|\to 0} f(pq) = 1$ where $|pq|$ = distance between p and q. The author considers the random variable $\eta = |D|^{-1} \iint_D y(p)dp$, i.e. the mean of a sample function $y(p)$ of $Y(p)$, and studies three methods of estimating the value assumed by η: (1) random sampling, which consists of choosing points $p_1, p_2, \cdots, p_n$ in D at random, assuming uniform distribution in D, and computing the estimate $\bar{\eta} = [y(p_1) + \cdots + y(p_n)]/n$;

(2) stratified sampling, which consists in subdividing D in disjoint regions $D_1, D_2, \cdots, D_n$, choosing $p_1, p_2, \cdots, p_n$ at random so that p_i is drawn from a uniform distribution in D_i, and computing $\bar{\eta}$; (3) systematic sampling, which assumes that D can be decomposed into disjoint regions $D_1, \cdots, D_n$ which are congruent by translation, and consists in first obtaining p_1 at random from a uniform distribution in D_1, then obtaining p_i by translating p_1 into D_i, and computing $\bar{\eta}$. The quantity $E(\eta-\bar{\eta})^2$ is denoted by s_r^2, s_{st}^2, or s_{sy}^2, according to whether $\bar{\eta}$ is obtained by random, stratified, or systematic sampling. Explicit expressions are obtained for these three quantities, and it is shown that always $s_{st}^2 \leqq s_r^2$. Furthermore, the author states a sufficient condition for $s_{sy}^2 \leqq s_{st}^2$, but also exhibits a class of processes for which either $s_{sy}^2 < s_{st}^2$ or $s_{sy}^2 > s_{st}^2$ depending on the choice of scale parameters in the plane.

Z. W. Birnbaum (Seattle, Wash.)

417:

Coleman, B. D. On the strength of classical fibres and fibre bundles. J. Mech. Phys. Solids **7** (1958), 60–70.

In this paper the author discusses the distribution of tensile strengths of fibres, whose strength is independent of the rate of loading (such fibres are called "classical"). Both single fibres and fibre bundles are considered. In the case of single fibres, it is shown that the tensile strength of long classical fibres should follow the Weibull distribution. This is done by treating the long fibre as if it is a long chain made up of many pieces of "unit" length and using the weakest link idea. If one assumes that the c.d.f. $G(x)$ of strengths X of unit lengths is identically zero for $x \leqq 0$ and approaches zero as some positive power of x as $x \to 0^+$, one is led to a type 3 asymptotic distribution of smallest values (a particular case of which is the Weibull distribution). Having treated the single fibre case, the author considers briefly the tensile strength distribution of a fibre bundle composed of a very large number of fibres, each following a common Weibull distribution. The principal result is that the ratio of the tensile strength (units of force at break per unit initial area) of a bundle to the mean tensile strength of the constituent fibres decreases as the coefficient of variation of the distribution of strengths of individual fibres increases.

Benjamin Epstein (Stanford, Calif.)

NUMERICAL METHODS

See also 57, 215, 300, 313, 480, 631, 632, 633.

418:

Cerulus, F.; and Hagedorn, R. A Monte-Carlo method to calculate multiple phase space integrals. I, II. Nuovo Cimento (10) **9** (1958), supplemento, 646–677.

This paper is concerned with the calculation of so-called "phase space" integrals of the form

$$(1)\quad \rho n(E, \mathbf{P}) = (2\pi\hbar)^{-(3n-3)} \int d\mathbf{p}_1 \cdots d\mathbf{p}_n \times \delta\Big(\mathbf{P} - \sum_{i=1}^{n} \mathbf{p}_i\Big)\delta\Big(E - \sum_{i=1}^{n} \sqrt{(p_i^2+m_i^2)}\Big).$$

These integrals arise in various physical problems, in particular in problems of multiple particle production. The authors choose to evaluate the integrals by the Monte Carlo method; they make a good case for this choice. The first paper gives the method, the second is largely concerned with evaluation of error in the procedure.

R. R. Coveyou (Oak Ridge, Tenn.)

419a:

★**Martinot-Lagarde, André. Quasi-dérivée.** Actes des colloques de calcul numérique, Caen, 1955; Dijon, 1956, pp. 25–36. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 77, Paris, 1958. vi+144 pp. 2105 francs.

419b:

★**Martinot-Lagarde, André. Sur une interprétation expérimentale de la dérivée.** Actes des colloques de calcul numérique, Caen, 1955; Dijon, 1956, pp. 37–42. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 77, Paris, 1958. vi+144 pp. 2105 francs.

These papers were summarized in the course of a later paper by the author [C. R. Acad. Sci. Paris **246** (1958), 690–692; MR **20** #2405]. It will therefore suffice to mention two propositions that he annexes to his discussion of a kind of "quasi-differentiability" for functions only known empirically. Let $f(x)$ be real-valued and continuous for $a \leqq x \leqq b$, let $r(x, x+h)$ denote $\{f(x+h)-f(x)\}/h$, and for $h>0$ let $W(h)$ equal the maximum of $|r(x,x+h)-r(y,y+h)|$ for $a \leqq x \leqq y \leqq x+h \leqq y+h \leqq b$. The author proves, by elementary arguments, that for $f(x)$ to be uniformly differentiable on (a, b) it is (i) necessary that $W(h) \to 0$ $(h \to 0)$ and (ii) sufficient that $W(h) = O(h)$. He notes that (ii) is a corollary of a much stronger theorem given by M. Zamansky [Ann. Sci. École Norm. Sup. (3) **66** (1949), 19–93; MR **11**, 27; p. 72, théorème 13].

H. P. Mulholland (Exeter)

420:

★**Buck, R. Creighton. Linear spaces and approximation theory.** On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 11–23. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

This introductory note to a book not now available to the reviewer is a concise and illuminating account of the kind of understanding to be obtained by viewing problems in approximation of functions—best uniform, best p'th power, or whatever—as a part of the geometric-algebraic pattern now fashionably called functional analysis.

M. M. Day (Urbana, Ill.)

421:

★**Walsh, Joseph L. On extremal approximations.** On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 209–216. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

A brief review of certain non-asymptotic aspects of the theory of polynomial approximation, emphasising results recently obtained by the author and others concerning the structure of extremal polynomials. The concepts of juxtapolynomial, underpolynomial, and infrapolynomial

are defined and discussed. Proofs are given for only one or two rather immediate results. There are a number of references to the literature.

J. H. Curtiss (Providence, R.I.)

422:

★Miller, J. C. P. Extremal approximations—a summary. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 329–340. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

This summary of several papers on numerical approximation deals mainly with practical applications of polynomial approximation. The author discusses the theorem of Weierstrass and the cardinal function of Whittaker, using this to demonstrate the effects of smoothing of an irregular table on the functional values and their derivatives, and possible applications to numerical quadrature. Chebyshev expansions are considered for purposes of approximation, the development of efficient interpolation formulae, and for use with automatic computers, and several methods are suggested for calculating the coefficients. Other possibilities, such as rational approximation and continued fractions, are mentioned briefly, and there is a useful bibliography of thirty-eight relevant publications.

L. Fox (Oxford)

423:

Erdős, P. Problems and results on the theory of interpolation. I. Acta Math. Acad. Sci. Hungar. **9** (1958), 381–388.

If $\begin{pmatrix} & x_1^{(1)} & \\ x_1^{(2)} & & x_2^{(2)} \\ \cdot & \cdot \quad \cdot & \cdot \end{pmatrix}$ is a triangular matrix of real numbers in $(-1, 1)$ and the unique polynomial of degree at most $n-1$ which assumes the values y_k at $x_k^{(n)} (1 \leq k \leq n)$ is given by $\sum_{k=1}^{n} y_k l_k(x)$, the author proves the following: Let ε and A be any given numbers and let $n > n_0 = n_0(\varepsilon, A)$; then the measure of the set in $x(-\infty < x < \infty)$ for which (1) $\sum_{k=1}^{n} l_k(x) \leq A$ holds is less than ε.

Stronger theorems are stated but not proved. For example, the measure of the set for which (1) holds for sufficiently large n is less than $c(A)/\log n$.

S. Macintyre (Aberdeen)

424:

Greville, T. N. E. The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations. SIAM Rev. **1** (1959), 38–43.

This is a mainly expository paper concerning the generalized inverse of an arbitrary rectangular matrix. This notion was investigated by R. Penrose [Proc. Cambridge Philos. Soc. **51** (1955), 406–413; MR **16**, 1082]. The author points out that it was introduced by E. H. Moore in his *General analysis* [Mem. Amer. Philos. Soc., vol. 1, Philadelphia] in 1935. In the reviewed paper an elementary presentation of the pseudoinverse and of its properties (with short proofs) is given. An application to the numerical solution of a general system of linear algebraic equations is added.

M. Fiedler (Prague)

425:

★Gouarné, René. Résolution rapide des systèmes d'équations linéaires. Actes des colloques de calcul numérique, Caen, 1955; Dijon, 1956, pp. 127–144. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 77, Paris, 1958. vi+144 pp. 2105 francs.

The paper is concerned with the solution of systems of linear equations and consists of two parts. In the first part, the author describes a geometrical representation of matrices which enables him to calculate determinants and thus characteristic polynomials. In the second part, he generalizes gaussian elimination by "pivoting" about a minor of arbitrary order of the matrix instead of about a minor of order 1.

H. H. Goldstine (Yorktown Heights, N.Y.)

426:

Goldstine, H. H.; and Horwitz, L. P. A procedure for the diagonalization of normal matrices. J. Assoc. Comput. Mach. **6** (1959), 176–195.

The so-called Jacobi diagonalization algorithm for symmetric matrices uses 2-dimensional rotations in planes spanned by 2 coordinate axis in order to diminish the sum τ^2 of the squared off-diagonal elements by an appropriate sequence of such rotations. The algorithm has been discussed very successfully by H. H. Goldstine. In this paper the authors introduce an analogous algorithm for the diagonalization of normal matrices; this is the most general case of matrices which can be diagonalized by unitary transformations. Furthermore normal matrices are important in quantum mechanics where they represent a set of simultaneously measurable observables. The paper is based on a thorough discussion of the decrease of τ^2 during a 2-dimensional unitary rotation and of the conditions that such a decrease takes place. Moreover, rules for an optimal choice of the parameters of the rotation are given in order to diminish τ^2 in each step as much as possible. In one exceptional case the given normal matrix has to be modified because the general algorithm would not lead to a convergent procedure.

E. Stiefel (Zürich)

427:

Ostrowski, A. M. On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors. I, II. Arch. Rational Mech. Anal. **1** (1958), 233–241; **2** (1958/59), 423–428.

Let A be a real symmetric matrix and let $\mu(x) = (x, Ax)/\|x\|^2$ be the corresponding Rayleigh quotient, where (x, y) is the usual inner product and $\|x\| = (x, x)^{\frac{1}{2}}$. Given a fixed vector $x_0 \neq 0$ let $\{x_k\}$ be the sequence of vectors generated by the formulas

$$\lambda_k = \mu(x_{k-1}), \quad x_k = (A - \lambda_k I)^{-1} x, \tag{1}$$

or by the formulas

$$\lambda_k = \mu(x_{k-1}), \quad x_k = (A - \lambda_k I)^{-1} x_{k-1}. \tag{2}$$

The present paper gives a careful analysis of the convergence of these two methods. It is proved that in the first case the convergence is quadratic and that in the second case it is cubic. Useful asymptotic formulas are given. The results are also valid for hermitian metrices.

In the second paper these results are extended so as to give further asymptotic formulas and to give methods of accelerating the convergence. Numerical examples are given.

M. R. Hestenes (Los Angeles, Calif.)

428:

Caldwell, George C. A note on the downhill method. J. Assoc. Comput. Mach. **6** (1959), 223–225.

The "downhill" method to determine a root of an analytic equation $f(z)=0$ is based on a "minimum-modulus" principle: If

$$f(z) = R(x, y)+iJ(x, y)$$

is analytic in a domain D of the z-plane, then

$$W(x, y) = |R(x, y)|+|J(x, y)|$$

has a minimum at an inner point of D if and only if this point is a zero of $f(z)$. {Actually $f(z)$ need not be analytic and a more general principle can easily be proved without calculations; see e.g. C. Caratheodory, *Funktionentheorie I* [Birkhauser, Basel, 1950; MR **12**, 248], p. 134 (§ 136); cf. p. 136 (§ 139) where a hint to numerical application in the sense of the "downhill" method is given in the case of the ordinary "minimum-modulus" principle. Similar proofs are given in the well-known books by Ahlfors and Markuševič.—Reviewer's remark.} J. A. Ward [J. Assoc. Comput. Mach. **4** (1957), 148–150; MR **19**, 1082] has proposed a step-by-step trial and error procedure to find a sequence of points (x, y) for which the corresponding values $W(x, y)$ decrease. In the present paper a method is indicated by which for a given (x, y) increments (h, k) can be found such that $W(x+h, y+k) < W(x, y)$.

H. Schwerdtfeger (Montreal)

429:

Remez, E. Ya. Sur quelques questions concernant la structure des formules de quadratures qui peuvent servir pour l'évaluation bilatérale numérique des solutions des équations différentielles. Ukrain. Mat. Ž. **10** (1958), 413–418. (Russian. French summary)

Let us consider an integration formula of the form

$$(4)\qquad \int_{x_r}^{x_n} f(x)dx = (x_n-x_r)\sum_{i=s}^{n} k_i f(x_i)+R$$
$$(r < n,\ s < n),\quad R = cf^{(q)}(\xi).$$

When the integration is forward, that is (3) $x_0 < x_1 < x_2 < \cdots$, the author proves theorem I: If in a quadrature formula of the form (4), where the x_i satisfy inequalities (3) and the coefficient c in the error expression is negative, then $k_n>0$; if conversely $c>0$, then $k_n\leqq 0$.

For backward integration, that is, for (12) $x_0 > x_1 > x_2 > \cdots$, the author proves theorem II: If we have a quadrature formula of form (4), where the x_i satisfy (12), then in the case of even q (q is the order of the derivative of the remainder term) we must have $k_n>0$ for $c>0$ and $k_n\leqq 0$ for $c<0$; but in the case of odd q, $k_n>0$ for $c<0$ and $k_n\leqq 0$ for $c>0$.

A. H. Stroud (Madison, Wis.)

430:

Morrison, David. Numerical quadrature in many dimensions. J. Assoc. Comput. Mach. **6** (1959), 219–222.

Let $f(x_1, \cdots, x_n)$ be a function defined on the hypercube $-a\leqq x_i\leqq a$ $(i=1, \cdots, n)$. With $i_1\leqq \cdots \leqq i_k\leqq n$ define the k-dimensional trace of f, denoted $P(x_{i_1}, \cdots, x_{i_k})f$, as the function obtained from f by setting all variables not among $x_{i_1}, \cdots, x_{i_k}$ equal to zero. The following result is stated.

Theorem: If f is a sum of functions of d variables, i.e.,

$$f(x_1, \cdots, x_n) = \sum g_{i_1, \cdots, i_d}(x_{i_1}, \cdots, x_{i_d}),$$

then f may be written in terms of its traces as

$$f(x_1, \cdots, x_n) = \binom{d-n}{0}\sum P(x_{i_1}, \cdots, x_{i_d})f$$
$$+\binom{d-n}{1}\sum P(x_{i_1}, \cdots, x_{i_{d-1}})f$$
$$+\binom{d-n}{2}\sum P(x_{i_1}, \cdots, x_{i_{d-2}})f+\cdots$$
$$+\binom{d-n}{d-1}\sum P(x_{i_1})f+\binom{d-n}{d} Pf.$$

The summations are over all integers i_j with $1\leqq i_j\leqq n$ and $i_j < i_{j+1}$.

The author uses this decomposition to derive a numerical quadrature formula for the hypercube of degree five for a particular kind of symmetrical weight function. A specialization gives a formula reported by Hammer and Stroud.

P. C. Hammer (Madison, Wis.)

431:

Wolfe, J. M. An adjusted trapezoidal rule using function values within the range of integration. Amer. Math. Monthly **66** (1959), 125–127.

In this note the author shows that, for functions $y(x)$ such that $y''(x)$ does not change signs, the "adjusted trapezoidal method" of numerical integration is better than the trapezoidal. The reviewer previously published a note [Math. Mag. **31** (1957/58), 193–195; MR **20** #6191] which establishes this result under somewhat more general conditions. It is perhaps better to term the method the midpoint method and, as such, it may be considered an "open" Newton Cotes formula or a 1-point Gauss formula. Bounds for integrals may be established using both trapezoidal and the midpoint method as the author notes and as was noted in the paper of reference.

P. C. Hammer (Madison, Wis.)

432:

Ceschino, F.; et Kuntzmann, J. Impossibilité d'un certain type de formule d'intégration approchée à pas liés. Chiffres **1** (1958), 95–101.

The authors show that it is impossible to improve the Adam's numerical method for solving $y' = Y(t, y)$, $y(0)=y_0$ under stated conditions of stability and order of polynomial accuracy within the class of formulas of form

$$(1)\qquad y_{p+1} = \sum_{j=0}^{q} \alpha_j y_{p-j}+h\sum_{j=0}^{q} \alpha_j'\Delta^j Y_p.$$

The stability condition required is that the modules of roots of the equation $r^{q+1}=\sum_{j=0}^{q} \alpha_j r^{q-j}$ not exceed unity. The order condition required is that (1) be an identity when y_r is replaced by $P_{q+2}(rh)$ and Y_r is replaced by $P_{q+2}'(rh)$ where $P_n(x)$ is an arbitrary polynomial of degree n and $P_n'(x)$ is its derivative.

Next, certain implicit formulas are derived which satisfy stability requirements. The character of $Y(t, y)$ does not enter these arguments, the relations derived are statements concerning polynomials.

P. C. Hammer (Madison, Wis.)

433:

Korganoff, A. Sur des formules d'intégration numérique des équations différentielles donnant une approximation d'ordre élevé. Chiffres **1** (1958), 171–180.

In this paper the author considers formulas for numerical integration of the first order differential equation

$y' = f(x, y)$ based on the Hermite interpolation formula and on the Gauss quadrature formulas. Calculations are made on $y' = ay$ with $a = 1$ and $a = -1$ to compare the Gauss formula with Milne's formula. The reviewer and Hollingsworth previously published a paper demonstrating the use of Gauss quadrature for the same problem [cf. Math. Tables Aids Comput. **9** (1955), 92–96; MR **17**, 302].

The calculations made by the author show that the time needed for calculations using his algorithm was greater per step and about the same in total for comparable orders of accuracy as one of Milne based on a fixed step of advance. In the paper referred to above it was pointed out that trial calculations making use of linearity of the differential equations showed the Gauss quadrature formulas in a better comparable position.

P. C. Hammer (Madison, Wis.)

434:

Budak, B. M.; and Gorbunov, A. D. Stability of calculation processes involved in the solution of the Cauchy problem for the equation $dy/dx = f(x, y)$ by multipoint difference methods. Dokl. Akad. Nauk SSSR **124** (1959), 1191–1194. (Russian)

The author considers an initial value problem $dy/dx = f(x, y)$. The values of y_k at $x_k = x_0 + kh$ are determined by selected finite difference formulas

$$F_k \equiv \sum_{i=0}^{m} \alpha_i y_{k-i} - h \sum_{i=0}^{n} \beta_i f_{k+l-i} = 0 \quad (f_j = f(x_j, y_j))$$

together with some continuously differentiable function $g(x, h)$ which gives the required starting values for the difference equations. Let R be a method of certain precision for computing $y_k{}^*$ from $F_k = 0$, $y_i = g(x_i, h)$. It is said that the method (F, g, R) converges if $y(x_k) - y_k{}^* \to 0$ as $h \to 0$. The theorems of the paper stated without proof contain the following results.

(a) The method (F, g, R) converges if the method (F, g) (assuming no round-off error) converges and $(-1/h)\int^x F_k d\xi \to 0$ as $h \to 0$ uniformly in k, x, and θ. Here x is chosen in a suitable interval; θ is the difference between the actual number and its numerical representation by a finite fraction. The bound θ_h on θ converges to zero no faster than some positive degree of h.

(b) A necessary condition that (F, g, k) converges is that $(-1/h)F_k \to 0$ uniformly in k and θ as $h \to 0$, $\theta = O(h^t)$, $t > 0$.

(c) The condition of (b) is sufficient for convergence provided either (i) $\theta_h = o(h)$ or (ii) the absolute values of the roots of $\sum_{i=0}^{m} \sum_{j=0}^{i} \alpha_j \lambda^{m-1-i} = 0$ $(m > 1)$ are less than 1.

C. Masaitis (Havre de Grace, Md.)

435:

Wilf, Herbert S. An open formula for the numerical integration of first order differential equations. II. Math. Tables Aids Comput. **12** (1958), 55–58.

In der ersten Mitteilung [Math. Tables Aids Comput. **11** (1957), 201–203; MR **19**, 884] entwickelte der Verf. eine Formel zur numerischen Integration von Anfangswertaufgaben $y' = f(x, y)$, $y(x_0) = y_0$. Diese zweite Mitteilung enthält zugehörige Konvergenz- und Stabilitätsuntersuchungen sowie eine entsprechende Formel höherer Ordnung.

J. Schröder (Hamburg)

436:

Greenspan, Donald. On the numerical solution of Dirichlet-type problems. Amer. Math. Monthly **66** (1959), 40–46.

As in his earlier paper [Math. Tables Aids Comput. **11** (1957), pp. 150–160; MR **19**, 772] the author develops a bound on the difference between the exact solutions of boundary value problems involving the differential equations $U_{xx} + U_{\rho\rho} - (K/\rho)U_\rho = 0$, where K is a constant, and the solutions of certain finite difference equations. The error bounds are obtained by the application of the methods of Gerschogrin [Z. Angew. Math. Mech. **10** (1930), 373–382] and involve bounds on the third and fourth partial derivatives of the unknown solution.

D. M. Young, Jr. (Austin, Tex.)

437:

Durand, Emile. Sur les solutions élémentaires numériques du problème de Dirichlet dans le plan. Chiffres **1** (1958), 103–120.

For a finite region in two dimensions, an elementary solution is that solution of Laplace's equation which takes the value zero at all points of the boundary except one. Its value at this point is taken as unity. The method which is proposed for finding this elementary solution is that of covering the region with a square mesh, writing a finite difference approximation to the partial differential equation and solving the resulting set of linear simultaneous equations by iteration. For a semi-infinite region, an elementary solution is obtained by combining an appropriate analytical solution with that obtained by solving the set of simultaneous equations which approximate the differential equation. A linear combination of these elementary solutions is then used to find the solution of a specific problem.

Several examples are included in the paper. It should be noted that the greater part of the paper is devoted to finding the elementary solutions for semi-infinite regions.

M. Lister (State College, Pa.)

438:

Hochstrasser, U. W. Numerical experiments in potential theory using the Nehari estimates. Math. Tables Aids Comput. **12** (1958), 26–33.

Zeev Nehari [*Proceedings of the conference on differential equations*, pp. 157–178, Univ. of Maryland Book Store, College Park, Md., 1956; MR **18**, 602] has given estimates of the error committed by replacing infinite orthogonal expansions by finite ones for a number of Dirichlet problems. This paper reports the results of some numerical experiments which were designed to investigate how close these estimates are to the actual errors in certain particular examples. In the experiments an irregular pentagon inscribable in a circle was used as the region in which the Dirichlet problem was defined. For this region the following problems were considered: (a) the boundary values given by

$$U(x, y) = \text{Re}\,(\cos z) = \text{Re}\,[\cos(x+iy)] = \cos x \cosh y;$$

(b) the boundary values given by

$$U(x, y) = \text{Re}\left(\frac{1}{z+4}\right) = \frac{x+4}{(x+4)^2+y^2};$$

and (c) the boundary values are equal to one on one side and zero on all others.

The set of functions used to approximate the solution for cases a and b consisted of the harmonic polynomials obtained by orthonomalizing 1, Re z, Im z, Re z^2, Im z^2, $\cdots$ on the region. In addition to the preceding, the harmonic measure which has the value 1 on the arc of the circumscribed circle, cut out by the side of the pentagon with boundary value 1, and zero on the rest of the circle form the basic set for case c.

The number of harmonic functions in the basic set was varied from 3 to 11 in cases a and b and from 4 to 12 in case c.

For the numerical examples chosen, the Nehari estimate gave considerably better bounds for the error in the interior of the region than the maximum modulus principle The Nehari estimate was large with respect to the actual error by a factor of from 5 in some cases to almost 100 in others.

This paper does not seem to imply that the Nehari estimates can be sharpened, i.e., through different choices of boundary conditions the actual errors might be nearer to the Nehari estimates.

G. W. Evans, II (Menlo Park, Calif.)

439:

Sobolev, S. L. Remarks on the numerical solution of integral equations. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **11** (1957), no. 4 (23), 5–30.

Translation of a Russian original [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 413–436; MR **18**, 322].

440:

Štykan, A. B. Graphical solution of differential equations with leading argument for various types of initial condition. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 193–206. (Russian)

The author considers equations of the form $y'(x)=y(x+a(x))$, and discusses various approximate techniques for tracing out the solution, using ideas similar to those employed in the theory of ordinary differential equations.

R. Bellman (Santa Monica, Calif.)

441:

Kreĭnes, M. A.; and Kiškina, Z. M. Approximation by functions of the fifth nomographic order. Dokl. Akad. Nauk SSSR **125** (1959), 262–265. (Russian)

442:

Holberg, Karl; and Jensen, Jens R. Table of $W=Z/(1+Z)$ for complex numbers. Acta Polytech. Scandinav. **257** (1959), v+144 pp. (1 plate)

Cette table est destinée au calcul de: $W=Z/(1+Z)$ [ou de: $V=1/(1+U)$ avec $U=1/Z$] pour Z complexe. Son but essentiel est de remplacer les calculs graphiques dans l'étude de systèmes asservis en télécommunications; les notations un peu spéciales sont empruntées à cette technique.

La table fournit: 20 Log $|W|$ (arrondi à la première décimale) et Arg W en degrés (arrondi au degré), pour: 20 Log $|Z|=-22.0(0.2)26.0$; Arg $Z=-178°(2°)0°$.

Des formules approchées sont données lorsque Z sort des limites de la table. Par contre, aucun renseignement n'est fourni pour une interpolation éventuelle.

La disposition adoptée est assez claire, mais l'impression laisse à désirer.

G. Brillouet (Nantes)

COMPUTING MACHINES

See also 8, 387, 480, 637, 638, 639, 640.

443:

★**Mandl, Matthew. Fundamentals of digital computers.** Prentice-Hall, Inc., Englewood Cliffs, N.J., 1958. xi+297 pp.

The technician who is familar with radio electronics and wishes to enter the computer maintenance field finds many of the same components in use, but under very different conditions.

The author reviews the characteristics of vacuum tubes, diodes and transistors, pointing out the distinctive uses of these devices in computers. He describes in some detail other elements not found in the radio field, such as the magnetic core and the cryotron, explaining their use as bistable elements. An attempt is made to familiarize the technician with computer signal forms, although this reviewer found the definitions confusing and inconsistent.

Chapters on basic circuits, calculation circuits, special circuits, computer arithmetic and programming are designed to give the reader an outline of computer logic, coding and application of binary arithmetic to computer design, without formal use of Boolean algebra.

A chapter on commercial computers lists and describes briefly about a dozen of these devices.

A chapter on maintenance factors and an appendix on color codes for resistors and capacitors conclude the volume.

The book contains a great deal of useful data for the maintenance man, though it might profit from more care in definitions and in certain explanations. This reviewer found the example of excess-three subtraction confusing, even though he originated that code twenty years ago, and feels fairly familiar with it.

G. R. Stibitz (Cambridge, Vt.)

444:

Lyubčenko, G. G. A method of selecting logical operations and mechanisms for effecting them for numerical computing machines. Ukrain. Mat. Ž. **10** (1958), 375–388. (Russian. English summary)

The author presents a discussion of a method of selection (by a computing machine) of sets of logical operations suitable for the synthesis of (arbitrary) logical expressions. The suitability of a set depends upon such factors as the minimizations of complexity, time, and energy spent (by the particular machine) on the average in such syntheses.

R. M. Baer (Berkeley, Calif.)

445:

★**Andree, Richard V. Programming the IBM 650 magnetic drum computer and data-processing machine.** Henry Holt and Co., Inc., New York, 1958. vi+109 pp. (5 plates) \$2.95.

This is a good practical tutor in the art of programming for the I.B.M. 650, containing a large number of useful exercises. All the usual techniques are introduced at appropriate stages. The chapters on the Symbolic Optimal Assembly Program and on Compilers are intended to be supplemented by I.B.M.'s SOAP and FORTRANSIT manuals.

A. M. Duguid (Providence, R.I.)

446:

Newell, Allen; Shaw, J. C.; and Simon, H. A. Chess-playing programs and the problem of complexity. IBM J. Res. Develop. **2** (1958), 320–335.

The authors compare recent attempts to describe intricate mechanical procedures which search for and select appropriate moves in chess. In all cases these procedures are discussed or presented as feasible programs for general purpose digital computers. They begin with Shannon's analysis [Philos. Mag. (7) **41** (1950), 256–275; MR **11**, 543] of the game as a large tree of valid positions with an appropriate selection of alternative moves, an appropriate exploration of continuations to a suitable depth, an evaluation of each continuation by a suitable criterion, and a selection method of a next move by means of an appropriate criterion. Shannon proposed searching all continuations to a fixed depth, evaluating by a single numerical function of stated position parameters, and selecting by a minimaxing procedure. The limitations in speed and storage capacity of present machines severely restrict the quality of the resulting play, not only by Shannon's procedure but by those further ones discussed, all of which essentially follow his analysis with occasional variations. These include programs by Turing for hand simulation [B. V. Bowden, *Faster than thought*, Pitman, London, 1953; MR **15**, 901; chap. 25], the Los Alamos machine program [Kister, Stein, Ulam, Walden and Wells, J. Assoc. Comput. Mach. **4** (1957), 174–177] and the IBM 704 program [Bernstein, Roberts, Arbuckle, and Belsky, A chess playing program for the IBM 704, Proc. 1958 Western Joint Computer Conference, May, 1958]. Turing's evaluation function was mainly based on 'material value' at the static conclusion of those continuations involving exchanges. His program was a weak player. The Los Alamos program restricted itself to a 6×6 board (no bishops, no castling, etc.), considered all continuations two moves deep, and included mobility as well as material value in its evaluating function. It could beat a weak player, required 12 minutes per move, and needed only 600 machine instruction words. Finally, the IBM 704 program was for a full 8×8 board, considered only 7 plausible moves, with all continuations to a depth of two moves, and used an evaluation function including the consideration of king defense and area control, as well as mobility and material. A novelty in the programming technique is a sequence of 'plausible move generators'. The program required eight minutes per move and some 7000 machine instruction words. It was beaten by a good player.

The program designed by the authors for the Rand Johnniac presents a number of notable advances. First of all it does not use a single evaluation number, but rather a vector of evaluations, accepting from a variable number of plausible moves the first achieving a number of goals whose priority can be changed in the program as the game proceeds. As in the Turing program, continuations are pursued until static positions with respect to the various goals are reached; and, as in the 704 program, a sequence of plausible move generating subroutines is used.

The second innovation is a powerful advance in the direction of flexibility. An automatic coding technique is used. Instead of programming in machine code, an interpretive routine permits the authors to program directly in an information processing command language of their own design [see Newell, Shaw, and Simon, Psych. Rev. **65** (1958), 151–166]. They can therefore analyze heuristically human motivations and evaluations with respect to various types of subgoals in the game, specify these and program them as subroutines in their information processing language. The language (IPL) itself is designed for easy handling of tree structures and ready modification of programs operating upon them. Thus, as the programmer sees his program beaten, he can question the victor and add a new goal subroutine to the program, which therefore does not really have a fixed size.

The authors estimate that their stated plan will need some 16,000 instruction words, both machine and IPL, and that each move might require from one to ten hours. A good part of the slowness is due to the translation time the machine needs in interpreting the IPL instructions.

The reviewer ventures to guess that not only will such limitations disappear with the design of a new generation of machines, but that the programs will be capable of simulating human intelligence in still another direction. The authors' program has the simulation built into it by the programmer. The same program, faced with the same position will always produce the same move. However, it is possible, using the very programming techniques invented by the authors, as well as stochastic learning models, to program the machine to store its experience in successive games so as to modify its own response program. Any example of this in a game as complex as chess must wait for the next generation of computing machines, further advances in programming techniques, and advance in the design of formal languages.

S. Gorn (Philadelphia, Pa.)

447:

Hildebrandt, Paul; and Isbitz, Harold. Radix exchange—an internal sorting method for digital computers. J. Assoc. Comput. Mach. **6** (1959), 156–163.

A new method, called radix exchange, is described for internal sorting on a binary digital computer. It is comparable with the methods of inserting and exchanging in that the working space required, in addition to that for the data and program, is negligible. It is superior to these methods in that the number of instruction cycles for sorting the first F integers by inserting (which is faster than exchanging) is proportional to F^2, whereas for radix exchange the number is approximately proportional to $F \log_2 F$. The number of cycles in merging is also $\sim F \log_2 F$, but considerable extra working storage is needed. Experimental tests on randomly arranged data showed that in the cases tested the method of radix exchange was much faster than inserting.

C. C. Gotlieb (Toronto, Ont.)

448:

Good, I. J. Could a machine make probability judgments? I, II. Computers and Automation **8** (1958), 14–16, 24–26.

The author answers the title question positively in an informal discussion. The article includes discussions of whether a computer can be programmed to think (yes); various theories "probability", "rational degrees of belief" and "credibilities"; computer programs that print probability judgments; and programs for playing chess. The author believes that computer programs will be supplemented by "partially random networks" in order to get machines that make probability judgements in a convincing sense.

J. McCarthy (Cambridge, Mass.)

449:

Good, I. J. How much science can you have at your finger-tips? IBM J. Res. Develop. **2** (1958), 282–288.

This is about information retrieval with and without artificial aids. There is a discussion of the information capacity of the brain compared with the amounts of published information of various kinds. It is pointed out that learning fundamentals and generalizations will help in the storage problem. Knowledge is compared with a kind of highly connected network and there are difficulties in classification where the network is not tree-like. The paper goes on to make many intriguing, if disconnected, suggestions about ways in which scientific communication can be improved and ends with a reminder that some of the greatest scientists did some of their best work in relative isolation. The paper has a far-ranging bibliography. *M. L. Minsky* (Cambridge, Mass.)

450:

Parker, E. T.; and Nikolai, Paul J. A search for analogues of the Mathieu groups. Math. Tables Aids Comput. **12** (1958), 38–43.

The authors describe a search by means of the UNIVAC Scientific Computer and the 1103A for groups similar to the remarkable quadruply transitive Mathieu groups of degrees 11 and 23. No such group with degree ≤ 4079 was discovered.

More precisely, the group G was required to have the following three properties: (1) G is transitive and of degree $p=2q+1$ with p and q primes; (2) G is simple but not cyclic; and (3) G is a proper subgroup of the alternating group of degree p.

There is a discussion of the method used for bringing the search for such a G within the cognizance of the digital computer and a very brief description of the code. Groups were rejected at the rate of about one group per second.

The authors conjecture that, except for $p=7$, 11, and 23, an unsolvable transitive group of degree $p=2q+1$ (with p and q primes) must be either the alternating or symmetric group. *D. H. Lehmer* (Berkeley, Calif.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 166, 172, 174, 175.

451:

Onicescu, O. Une mécanique des systèmes inertiaux. Une théorie de la gravitation. Une mécanique des petites distances. J. Math. Mech. **7** (1958), 723–740.

The author emphasizes need for a uniformly valid mechanics which departs from Newtonian form not only for high velocities but also for small distances. He formulates a Hamiltonian mechanic in which the counterpart of the Hamiltonian function is adjusted to allow for variable masses in Euclidean space. Postulates of invariance under change of frames of reference attached to the moving points and change of mass-energy scale lead to a definite system. The author discusses cases when his equations reduce, approximately, to those of Newtonian mechanics, to those of the relativistic theory of gravitation, and to a case considered appropriate for nucleons.

C. Truesdell (Bloomington, Ind.)

452:

Matschinski, Matthias. Mécanique correspondant aux équations fonctionnelles. C. R. Acad. Sci. Paris **248** (1959), 768–771.

In order to generalise the equations of mechanics beyond the scope of the ordinary differential equations, relations $x=f(t)$ ("minuscules") expressed by ordinary functions are supplemented by relations $X=\Phi(\theta(\tau), t)$ ("majuscules") expressed by aid of "functionals" (Fredholm, Volterra). Of the four possible types of differential equations only

$$g(X, X', X'', \cdots, t) = 0 \quad G(X, X', X'', \cdots, t) = 0$$

are considered. Introducing parameters $P=\Phi(\theta(\tau), t)$ the P' in their totality represent the "generalised velocity" which when put equal to a functional constant expresses the postulate of "generalised inertia". A further (quasi-Newtonian) postulate is

$$cP'' = f \equiv \text{generalised force.}$$

The application of this general theory to rheology is discussed in some detail in the following cases: $f=0$, $f=$function of, $f=$functional of the independent variables, $f=$function or functional linear in P.

E. B. Schieldrop (Oslo)

453:

Loiseau, Jean. La physique mathématique et la représentation de l'espace. Publ. Sci. Univ. Alger. Sér. A **4** (1957), 77–86.

In continuation of earlier work [*La mécanique rationelle dans un espace à 4 dimensions*, Publ. Sci. Tech. Ministère de l'Air, no. 270, Paris, 1952; MR **14**, 916] the author regards mathematical physics as a Newtonian mechanics in a 3-dimensional space H endowed with a euclidean connection. In order to obtain "observable relations" he considers H as being imbedded in a 4-dimensional Riemannian space. The resulting geometrical constructions are somewhat complicated, which seems to obscure any physical interpretation. *H. Rund* (Durban)

454:

Battaglia, Luana. Sul principio dell'effetto giroscopico nei solidi autoeccitati. Riv. Mat. Univ. Parma **8** (1957), 73–80.

The author obtains analytically a customary approximation for gyroscopic motion when, among other things, the velocity of rotation about the gyroscopic axis is large in comparison to other angular velocities.

C. R. De Prima (Pasadena, Calif.)

455:

Gutmann, Marcian. Quelques éléments de mécanique aréolaire. Bul. Inst. Politeh. București **19** (1957), 45–53. (Romanian. Russian and French summaries)

L'auteur démontre que pour les vitesses et les accélérations de surface (ou kepleriennes) on a dans la théorie du mouvement relatif des formules analogues à celles de Roberval et de Coriolis. *O. Bottema* (Delft)

456:

Gotusso, Guido. Sulla tenacia dei sistemi meccanici in movimento. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 693–700.

Author starts from a mechanical system of n degrees of

freedom with kinetic energy $T=\frac{1}{2}a_{hk}\dot{q}^h\dot{q}^k$ and writes Lagrange's equations as

$$a_{hk}\ddot{q}^h+[hl,\,k]\dot{q}^l\dot{q}^h = Q_k,$$

making use of tensor analysis notations, [ ] being Christoffel's symbols. Putting $\dot{q}_k=b^{ik}\omega_i$, where b^{ik} are given functions of q, the equations may be written

$$b^{hk}\dot{\omega}_h+C^{ijk}\omega_i\omega_j = Q^k.$$

The author considers moreover the more general case of an anholonomic system. If now ω_1 is large in comparison with the other ω_i, the system shows the imperturbability ("tenacia") which is well-known from the theory of the gyroscope, which occurs here as an example of a general theorem. *O. Bottema* (Delft)

457:

Jeffreys, Harold. The simple pendulum under periodic disturbance. Quart. J. Mech. Appl. Math. **12** (1959), 124–128.

The periodic motions of a rigid pendulum are considered when its point is given a forced horizontal oscillation of period τ, whose maximum acceleration ε is small compared with g, but the inclination of the pendulum to the vertical is not restricted. If the period T of small free oscillations exceeds τ there are three possible periodic motions, all of period τ: two of large amplitude and one small. As T is diminished the first two of these approach one another, until when T slightly exceeds τ they coalesce, at a motion whose amplitude is of order $\varepsilon^{\frac{1}{3}}$; and for smaller T these two motions are unreal. The elementary (linear approximation) solution for $T\neq\tau$ corresponds to two different branches of the general periodic solution.
T. M. Cherry (Melbourne)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 547, 548, 603, 605, 606.

458:

Stippel, H. Die Verwendung von Dipolsingularitäten bei der Berechnung heterogener Reaktoren. Acta Phys. Austriaca **12** (1958), 123–143.

459:

Klein, Martin J. Ehrenfest's contributions to the development of quantum statistics. I, II. Nederl. Akad. Wetensch. Proc. Ser. B **62** (1959), 41–62.

A survey of Ehrenfest's contribution to the understanding of the nature of photons, and their properties which were implied by Planck's radiation law, to the treatment of dissociating systems and the Gibbs paradox, and to the clarification of physical principles involved in Bose-Einstein and Fermi-Dirac statistics.
D. ter Haar (Oxford)

460:

Hartogh, C. D.; and Tolhoek, H. A. Cluster developments for Jastrow wave functions. II. Introduction of irreducible cluster functions. Physica **24** (1958), 875–895.

This is a continuation of Paper I (Physica **24** (1958), 721–741; MR **20** #5059]. In the previous paper a cluster expansion was developed for k-particle distribution functions in a large system of spinless fermions the ground state being described by a Jastrow wave function. The expansion is in terms of cluster integrals. In this paper the cluster integrals are expressed as sums of irreducible cluster functions, also generalizing the system to a mixture of fermions of different types. The method is analogous to the standard procedure used in statistical mechanics. *N. L. Balazs* (Chicago, Ill.)

461:

Hartogh, C. D.; and Tolhoek, H. A. Cluster developments for Jastrow wave functions. III. Expressions for the distribution functions and the energy; application to nuclear matter; expansions at low temperature. Physica **24** (1958), 896–909.

In this last paper of a series (see preceding review), the expansion procedure previously developed is generalized to include particles with spin and isobaric spin. The results are applied as follows: (1) a critical discussion is given concerning the consistency of variational approximations to wave functions using cluster expansions; (2) the grand partition function is found in terms of a cluster development for an imperfect Fermi or Bose gas.
N. L. Balazs (Chicago, Ill.)

ELASTICITY, PLASTICITY

See also 417, 452, 497, 593, 623.

462:

Stoppelli, Francesco. Sull'esistenza di soluzioni delle equazioni dell'elastostatica isoterma nel caso di sollecitazioni dotate di assi di equilibrio. II, III. Ricerche Mat. **7** (1958), 71–101, 138–152.

Continuing his definitive researches on the existence and uniqueness of solutions of the stress boundary value problem of finite hyperelasticity [same Ricerche **3** (1954), 247–267; **4** (1955), 58–73; **6** (1957), 11–26, 241–287; MR **17**, 554, 801; **19**, 901; **20** #2118], the author now concludes his investigation of the difficult case when the applied loads have an axis of equilibrium. The results are too technical to be stated in full here. The author shows that there are three cases, based upon the number and nature of the principal orientations of the body, to be distinguished. In one of these the situation is much as in the general case; if we require a solution which vanishes "in the strong sense" (defined by the author) for $\theta=0$, θ being the load parameter, then it exists, is unique, and may be developed in powers of θ. In another it is possible that a solution may fail to exist, or may exist but fail to possess a series expansion in θ. In the third, a solution always exists but may fail to be unique or fail to have a power series expansion. In the second and third cases, there may be solutions valid only for $\theta\geqq 0$ or only for $\theta\leqq 0$. Some comfort is brought by the fact that there are always at least two orientations such that nearby there are configurations of equilibrium.

The third part determines the behavior of solutions in the case when the load is unidirectional. If there is only one axis of equilibrium, there are infinitely many solutions of a certain type; the author determines a condition which individualizes the unique one that may be developed in a

series of functions vanishing with θ. If the loading is astatic, a solution does not exist unless a further condition is satisfied; under one further assumption, the author again finds means to individualize a regular solution.

C. Truesdell (Bloomington, Ind.)

463:

Hăimovici, Adolf. Quelques observations sur un phénomène d'hystérésis élastique, non-linéaire. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. 7 (1956), no 1, 9–13. (Romanian. Russian and French summaries)

Dans ses études d'hystérésis élastique, Volterra [Acta Math **35** (1912), 295–356] étudiait l'angle de torsion d'un fil, avec un coefficient constant K de proportionalité au moment M de torsion et un facteur additif héréditaire, L'A. propose d'ajouter aussi à K un tel facteur héréditaire (intégrale contenant M, du type $\int_{-\infty}^{t} f(t-\tau)M(\tau)d\tau$). Cela donne une équation intégrale dont l'existence et l'unicité de la solution sont assurées par le théorème du point fixe, sous certaines conditions.

M. Brelot (Paris)

464:

Brown, E. H. A theorem of maximum strain energy. J. Appl. Mech. **26** (1959), 73–76.

A new theorem for elastic structures obeying Hooke's law is proved and enunciated as follows: If the strain energy can be expressed in terms of given external loads and the positions of points constrained to have no displacement, then the expression will have a maximum value when the positions are such as make the constraining forces zero. The theorem is generalized to allow a wider class of constraints, such as the attachment of a second, rigid structure to the elastic structure under analysis, in such a way that the two have certain displacement components in common. The principle has been applied by E. H. Mansfield to certain problems in the bending of plates. (From the author's summary.)

E. H. Mansfield (Farnborough)

465:

★Diaz, J. B.; and Payne, L. E. Mean value theorems in the theory of elasticity. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 293–303. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

Mean value theorems for plane and three dimensional elasticity are obtained simultaneously by considering elastic equations in n dimensional real Euclidean space with coordinates $(x_1, x_2 \cdots, x_n)$. In each theorem the value of an elastic "quantity" such as component of displacement, strain or stress, evaluated at the origin $O=(0, 0, \cdots, 0)$ is expressed in terms of corresponding elastic quantities integrated over the surface (or the interior, as the case may be) of the sphere of centre O and radius $R>0$. Theorems are obtained for (i) displacement components in terms of the displacements, (ii) strain and stress components in terms of the displacements, (iii) stress components in terms of surface tractions, (iv) stress components in terms of normal surface displacement and normal surface stress.

A. E. Green (Newcastle-upon-Tyne)

466:

Garibaldi, Antonio Carlo. Su una proprietà di media caratteristica per l'equazione dell'elastostatica. Rend. Sem. Mat. Univ. Padova **27** (1957), 306–318.

The author generalizes the mean value theorem for linear elasticity due to Sbrana [Atti Accad. Ligure **9** (1952), 84–88; MR **15**, 179]. According to this theorem, the displacement field in a linearly and isotropically elastic body satisfies the spherical mean-value theorem

$$(*) \qquad s(P) = \frac{1}{4\pi\rho^2} \oint u(P, Q)d\sigma,$$

where $u(P, Q)$ is a certain linear combination of $s(Q)$ and of its tangential projection upon the surface at Q of a sphere with center at P and with radius ρ. The author proves that every vector summable in a domain T and satisfying (*) a.e. in T and for almost all ρ such that $0 \leq \rho \leq R$ is a solution of the differential equations a.e. in T_R. This result generalizes a theorem of Levi and Tonelli in the theory of the potential; T_R is defined as in that theorem. Thus Sbrana's mean value theorem is shown to be a characterizing integral formulation of classical linear elasticity.

C. Truesdell (Bloomington, Ind.)

467:

Mişicu, M. Représentation des équations de l'équilibre élastique par des fonctions monogènes de quaternions. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 458–470. (Romanian. Russian and French summaries)

The author studies elastic equilibrium by means of quaternion calculus. Boundary-conditions are also indicated for both the static and dynamic cases.

K. Bhagwandin (Oslo)

468:

Sparacio, Renato. Generalizzazioni del teorema di Castigliano. Rend. Accad. Sci. Fis. Mat. Napoli (4) **24** (1957), 145–151.

Forces $F_i^{(r)}$, $r=1, \cdots, n$, are applied to an elastic structure to produce stress σ_{ij} and strain ε_{ij}. Additional mechanical and thermal effects then produce additional strain ε_{ij}^*. The final displacement $u_i^{(r)}$ of the point of application of $F_i^{(r)}$ is given by

$$u_i^{(r)} = \frac{\partial L}{\partial F_i^{(r)}},$$

where

$$L = \tfrac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}dV + \int_V \sigma_{ij}\varepsilon_{ij}^* dV.$$

D. R. Bland (Manchester)

469:

Green, A. E.; and Spencer, A. J. M. The stability of a circular cylinder under finite extension and torsion. J. Math. Phys. **37** (1959), 316–338.

The non-linear elasticity problem of the finite extension and torsion of a solid circular cylinder of incompressible isotropic material has been solved by Rivlin; the present paper applies the perturbation method of Green, Rivlin, and Shield to consider the effects of a small deformation superposed on the finite deformation. Specifically, instability is analyzed by finding the loading conditions for which an adjacent equilibrium position exists. A particular strain energy function (Neo-Hookean) is chosen, and the condition for existence of an adjacent equilibrium position is obtained in the form of a transcendental equation, which is solved numerically for two loading conditions. The reduction to the linear elasticity case is also checked. The labor is considerable, and the results form a useful addition to the literature on non-linear elastic stability theory.

The authors use general tensors associated with a curvilinear coordinate system moving with the body. As a minor comment, the reviewer feels that there is no advantage over the use of conventional Lagrangian co-ordinates; also the equivalent variational formulation of the result would probably have been more compact.
C. E. Pearson (Cambridge, Mass.)

470:
Teodorescu, Petre P. Sur le problème plan de la théorie de l'élasticité dans le cas de certaines forces massiques quelconques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 481–489. (Romanian. Russian and French summaries)
The author establishes the following theorem: "Dans le cas du problème plan de la théorie de l'élasticité avec des forces massiques quelconques, le fonction $F=F(x, y)$ des efforts, ainsi que ses dérivées partielles, sont déterminées d'une manière univoque dans l'intérieur d'un domaine simplement connexe, si les charges de contour sont équilibrées par les forces massiques." The deduction is somewhat complicated, and explicit solutions are not presented. *K. Bhagwandin* (Oslo)

471:
Kaufman, R. N. Solutions of some boundary value problems of static theory of elasticity for a layer with a spherical cavity. J. Appl. Math. Mech. **22** (1958), 451–465 (327–337 Prikl. Mat Meh.).
This paper contains a purely formal treatment, within classical elasticity theory, of certain equilibrium problems for a body which is bounded by two parallel planes (infinite plate) with a spherical cavity. The bounding planes are subjected to zero shearing tractions and constant normal displacements (the plate is squeezed between two smooth punches of infinite extent). Various combinations of displacement and traction conditions for the spherical boundary are considered. The problem is reduced to the solution of an infinite system of linear algebraic equations. Numerical results are not included.
E. Sternberg (Providence, R. I.)

472:
Teodorescu, Petre P. Sur l'hypothèse des sections planes en résistance des matériaux. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 14, 41–46. (Romanian. French and Russian summaries)
As the author's results are properly summarized, we reproduce his summary below. "L'auteur s'occupe de l'hypothèse des sections planes pour les barres prismatiques surtout dans le cas de la flexion pure. Dans tous les traités classiques ⋯ on affirme que cette hypothèse est vérifiée par la théorie de l'élasticité. L'auteur démontre que l'hypothèse des sections planes n'est justifiée par la théorie de l'élasticité que si l'on néglige la contraction transversale (c'est à dire le coefficient de Poisson μ). A cette occasion, l'auteur fait diverses considérations sur la déformation de la section transversale des barres soumises à une flexion pure." *K. Bhagwandin* (Oslo)

473:
Čobanyan, K. S. The general problem of the bending of a rod composed of various materials. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 5, 121–127. (Russian. Armenian summary)
Bending cum torsion of a cantilever cylindrical beam composed of various materials and acted on by a transverse force applied at a point of the terminal cross-section is considered. The z-axis of the Cartesian reference frame, parallel to the generators of the surface of the beam, is arbitrarily located (does not necessarily pass through the reduced centroids of the cross-sections), while the x- and y-axis do not coincide with the reduced principal axes. With the cross-sections remaining plane after the deformation a set of Poisson equations for the stress functions in the regions with various elastic properties is obtained.
J. Nowinski (Madison, Wis.)

474:
Forsman, N. A. Stress concentrations in a stretched rod of circular cross-section. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 11, 25–30. (Russian)
The writer derives approximate formulas for the calculation of the stress concentrations in a stretched notched rod of circular cross section. The starting set of equations for the stresses is that derived by Heinz Neuber [*Kerbspannungslehre*, Springen, Berlin, 1937]. In the case of axial symmetry the stresses may be represented by means of equations containing the Laplacian of the Love function. The latter is represented in an integral form with the integrand expressed in terms of Bessel functions. The evaluation of the integrals is achieved by means of Simpson's rule with the approximate expression for the Bessel functions. The results are represented in forms of diagrams of stresses vs. the ratio t/R (t=the depth and R the radius of the notch).
M. Z. v. Krzywoblocki (Urbana, Ill.)

475:
Pârvu, A. Contribution à l'étude de la courbure des poutres. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 12, 19–22. (Romanian. French and Russian summaries)
The author establishes an equation for the mean deformation of a beam subject to the action of a uniformly distributed force. Neither the deduction nor the solutions presented are completely new. *K. Bhagwandin* (Oslo)

476:
Pârvu, Aurel. Contribution à l'étude de la torsion des poutres. I. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 13, 11–13. (Romanian. Russian and French summaries)
The author presents a method (Fourier series) for the determination of the stress function of beams subject to a torsional momentum. The solution is not new. Also, the author's expression for the two-dimensional Laplacian operator in polar co-ordinates is erroneous.
K. Bhagwandin (Oslo)

477:
Pârvu, Aurel. Contribution à l'étude de la torsion des poutres. II. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 14, 9–13. (Romanian. French and Russian summaries)
The author presents a method for analyzing the torsion in beams consisting of several isotropic media. He employs conformal mapping theory and Fourier series expansions. Some elementary cases are noted.
K. Bhagwandin (Oslo)

478:

Parsons, H. W.; and Leggett, D. M. A. The small deflection of a normally loaded square plate, elastically supported along its edges. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 1–10.

The authors study uniform pressure on an infinite uniform plate reinforced by a square mesh of uniform beams and supported at their intersections. Making the usual simplifying assumptions and using energy principles, they determine the important deflections and bending moments. *L. H. Donnell* (Chicago, Ill.)

479:

Stănescu, Cristian. Un problème du type mixte sur la flexion des plaques élastiques. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 411–421. (Romanian. Russian and French summaries)

The mixed fundamental problem in two-dimensional elasticity was discussed in detail by N. I. Muskhelishvili in his famed monograph [*Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, 1953; MR **15**, 370], by means of the complex variable method, for bodies occupying the half plane, the circle, and regions that may be mapped onto the circle by rational functions. In this paper the author discusses the analogous problem of transverse flexure of elastic isotropic thin plates whose middle planes occupy regions which may be mapped onto the circle by polynomials, by using the same method. The plate is clamped along certain segments of the boundary and free along the remaining segments. *Yi-Yuan Yu* (Brooklyn, N.Y.)

480:

★**Keller, Herbert B.; and Reiss, Edward L. Non-linear bending and buckling of circular plates.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 375–385. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The von Kármán equations for the nonlinear deflection of circular elastic plates are presented in finite difference form and are solved by iteration methods for various radially symmetric boundary conditions and loadings. The iteration procedure is adjusted to ensure convergence for arbitrarily large values of the loading to plate thickness parameter. The calculations were carried out on the UNIVAC. The method appears to be appreciably more powerful than the power series methods previously used for the same problems. The method is also successful in describing the rapid changes in the stresses that occur near the plate edge due to the boundary layer effect. *S. R. Bodner* (Providence, R.I.)

481:

★**Reissner, Eric. Rotationally symmetric problems in the theory of thin elastic shells.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 51–69. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

General problems in the theory of thin elastic shells require the determination of stresses and deformations as a function of two space co-ordinates. Rotationally symmetric problems have the property that stresses and deformations depend on one space co-ordinate only. This means that, for time-independent problems, one is concerned with ordinary differential equations rather than with partial differential equations. The present paper reviews a number of problems and solutions in the field of rotationally symmetric deformations of thin shells, all of them dealing with shells of revolution which are the most natural source of such problems. Consideration is given to linear and non-linear problems of the statics of shells of revolution, with particular emphasis on asymptotic solutions and edge effects. In addition, the problem of bending of pressurized curved tubes is formulated in considerable generality. (From author's summary) *L. S. D. Morley* (Farnborough)

482:

★**Bridgland, T. F., Jr.; and Nash, William A. Elastic deformations of a shallow shell in the form of an elliptic paraboloid.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 265–271. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The elastic deformations of a thin shell within the linearized theory can be derived from a stress function and the normal displacement of the middle surface for which Vlasov's equations hold if one refers to the lines of principal curvature. These two equations are applied here to the special case of a shell in the form of an elliptic paraboloid. To get a solution, the equations are first of all transformed formally into one single differential equation of a complex variable, as given by I. N. Vekua, and this leads to an integral equation of Volterra type. One problem, however, remains—and its solution for this special shell is the main purpose of the paper—to find the above mentioned transformation in an explicit manner. This involves two ordinary differential equations which are solved by elliptic integrals. *W. Schumann* (Zürich)

483:

Reissner, Eric. On influence coefficients and non-linearity for thin shells of revolution. J. Appl. Mech. **26** (1959), 69–72.

The paper is concerned with a non-linear formulation of the problem of rotationally symmetric deformations of thin elastic shells of revolution which are acted upon by edge forces and moments. Non-linear corrections, to the known results of linear theory, are determined for the edge displacements and rotations—the principal corrections are found to be those introducing a distinction between inward bending and outward bending.

The calculations are for cases where the thickness and curvature of the shell are such as to ensure that the stresses and deformations are effectively contained within a narrow edge zone of the shell. In particular, the validity of the corrections depends upon the absence (or negligible smallness) of distributed surface loads. *L. S. D. Morley* (Farnborough)

484:

Morley, L. S. D. An improvement on Donnell's approximation for thin-walled circular cylinders. Quart. J. Mech. Appl. Math. **12** (1959), 89–99.

The author uses nondimensional axial and circumferential coordinates x, θ and radial displacement w, equal

to the actual dimensions divided by the mean radius, and defines

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}.$$

By comparison with uncoupled Flügge equations for general types of radial displacement he shows that the limitations in applicability of the simplified Donnell equations for the radial load case can be removed by substituting $(\nabla^2)^2(\nabla^2+1)^2w$ for the $(\nabla^2)^4w$ term. He thus retains most of the advantages of the simplified equations without their limitations.

It is of interest to compare this with a previous result obtained by the reviewer [Proc. Fifth Internat. Cong. of Appl. Mech., pp. 66–70, Wiley, New York, 1938] which found that (in the author's symbols) substitution of $[(\nabla^2)^4 + 2\partial^6/\partial\theta^6 + \partial^4/\partial\theta^4]w$ for $(\nabla^2)^4w$ reduces maximum error for general radial loading to about the thickness-radius ratio. The new result contains these same terms plus many more, but, in the form in which the author presents it, it is probably easier to apply and seems to be even more accurate. The 1938 paper also gave simple general solutions for all types of static loading.

L. H. Donnell (Chicago, Ill.)

485:

Donnell, L. H. Effect of imperfections on buckling of thin cylinders under external pressure. J. Appl. Mech. **23** (1956), 569–575.

An approximate energy method solution is obtained for the nonlinear deformation of a hydrostatically loaded simply supported circular cylindrical shell. The analysis assumes the mode of initial imperfections to be the same as that of the load-produced deflections. Reasonable agreement with experiments is obtained for a physically postulated failure criterion and a chosen imperfection amplitude. Left open are questions of the stability and uniqueness of the solution in the nonlinear range.

S. R. Bodner (Providence, R.I.)

486:

Movsisyan, L. A. Some specific properties of anisotropic shells. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 4, 137–144. (Russian. Armenian summary)

Membrane states of stress in cylindrical, conical and spherical shells with fairly general anisotropy, the only plane of elastic symmetry being tangent to the middle surface of the shell, are considered. Trivial calculations show that, e.g., (a) a cantilever circular cylindrical shell suffers a twist if acted by a uniformly distributed load, the reversed load producing a reversed twist, (b) torsion of the shell (a) produces a radial displacement and a longitudinal stress with the rule of reversibility still valid, (c) hemispherical shell does not retain its spherical shape under the action of a uniform load, etc.

J. Nowinski (Madison, Wis.)

487:

Abramyan, B. L.; and Babloyan, A. A. Torsion of an anisotropic cylinder. Akad. Nauk Armyan. SSR. Dokl. **27** (1958), 269–275. (Russian. Armenian summary)

The torsion of a circular cylinder with cylindrical anisotropy the radial planes representing the planes of elastic symmetry are considered. The load is arbitrary but axially symmetrically distributed on the bases (B) and the curved surface (S) of the shaft. The solution of the differential equation for the stress function $\Phi(r, z)$ is represented by a series involving hyperbolic and Bessel functions, the coefficients of the series being determined from the integral static conditions on (B) and by comparing with the Fourier series into which the load on (S) is expanded. A numerical example is given.

J. Nowinski (Madison, Wis.)

488:

Čobanyan, K. S. Torsion of a composite shaft of variable diameter. Akad. Nauk Armyan. SSR. Dokl. **27** (1958), 139–144. (Russian. Armenian summary)

The axisymmetric problem of torsion of a shaft with variable cross-section composed of n various materials is reduced to a set of second order differential equations for the stress functions $\Phi_i(r, z)$ $(i = 1, 2, \cdots, n)$ with a boundary condition at the outer surface and a condition at the axis of the shaft, as well as two groups of conditions of continuity at $n-1$ surfaces bounding the regions of different materials. An illustrative example is given.

J. Nowinski (Madison, Wis.)

489:

Berg, B. A. Deformational anisotropy. J. Appl. Math. Mech. **22** (1958), 90–103 (67–77 Prikl. Mat. Meh.).

The author considers a particular quadratic theory of elasticity and determines the effect of initial stress arising from a large pure homogeneous strain upon the apparent elastic constants in a superposed small strain.

C. Truesdell (Bloomington, Ind.)

490:

Saunders, Herbert; and Paslay, Paul R. Inextensional vibrations of a sphere-cone shell combination. J. Acoust. Soc. Amer. **31** (1959), 579–583.

491:

★Hahne, H. V. Oscillations of a gas in an elastic cylindrical shell. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 753–760. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The frequency equation for the title problem is found as follows: (1) The formal solution by separation of the wave equation for the velocity potential Ψ in cylindrical coordinates (end condition $\Psi_x = 0$) yields an expression for the fluctuating pressure p_a at the wall $r = a$. (2) This p_a is used as the driver in the equations governing the displacements u, v, w of a thin cylindrical shell, the solutions being set up as double Fourier series with coefficients U_{km}, V_{km}, W_{km}. (3) The equation for W_{km} and another one, resulting from $\Psi_r = w_t$ at $r = a$, give infinitely many linear equations $(m = 1, 2, \cdots)$ for the expansion coefficients C_{kn}, k fixed. Here C_{kn} is the coefficients of $J_k(\alpha_{kn}r/a)\cos(n\pi x/L)$ times trigonometric factors in $k\varphi$ and ωt in the expansion of Ψ.

For purely circumferential $(n = 0)$ or for purely longitudinal $(k = 0)$ modes the frequency equation can be written explicitly; in the first case it permits the direct calculation of ω_{jk} vs. k for a particular example, which shows but a small coupling effect owing to the assumed smallness of h/a.

G. Kuerti (Cleveland, Ohio)

492:

Dun, Min-De. On the stability of elastic plates in a supersonic stream. Soviet Physics. Dokl. **120** (3) (1958), 479–483 (726–729 Dokl. Akad. Nauk SSSR).

The author studies the linear integro-differential equation governing the (small) deflection of an oscillating thin elastic plate in a supersonic inviscid stream. It is assumed that linearisation is valid and that the aerodynamic forces are the same as if the plate were of infinite span. By use of the Laplace transform he obtains the general solution for the deflection in closed (but complicated) form involving four arbitrary constants, so that the boundary conditions on the plate lead to the characteristic equation for the complex frequency. He believes that the determination of the frequency, and hence the stability boundaries from this equation by iterative methods is much simpler than the application of Galerkin's method to the original equation [see H. C. Nelson and H. J. Cunningham, N.A.C.A. Rep. 1280 (1956)], but no numerical results are presented. *H. C. Levey* (Nedlands)

493:

Capriz, Gianfranco. Alcune osservazioni sulla instabilità di una trave sollecitata a torsione. Riv. Mat. Univ. Parma **8** (1957), 145–160. (English summary)

The behaviour of an elastic rod exposed to twist has been studied by various authors taking as a model the case of a rod exposed to end-thrust. However, there arise differences as soon as the linear domain is passed. In the case of the end-thrust there exist non-trivial solutions for any load greater than critical, in the case of twist such solutions exist only corresponding to critical values of the twisting couple. A. Signorini [Ann. Mat. Pura. Appl. (4) **30** (1949), 1–72; **39** (1955), 147–201; MR **11**, 756; **18**, 246] has indicated a method to treat such problems by developing the characteristic quantities in series of a certain parameter λ. This method allows the splitting in an infinite succession of steps of any problem in the field of non-linear elasticity. The first step is the solution of the classical linearized equations, each following step takes account of a higher power of λ.

In the present note the author adapts this method to the actual problem. By the third step appears a system of equations which admits non-trivial solutions only corresponding to critical values of the twisting couple.

H. Bremekamp (Delft)

494:

Biot, M. A. The influence of gravity on the folding of a layered viscoelastic medium under compression. J. Franklin Inst. **267** (1959), 211–228.

The author extends his previous work [Proc. Roy. Soc. London Ser. A **272** (1957), 444–454; MR **19**, 1113] by the inclusion of a constant gravitational field acting parallel to the normals to the layer. An additional constant term is introduced into the equation connecting the pressure applied to the layer and the wavelength of the distortion. In the particular case of a viscous layer lying on a heavy viscous medium, the dominant wavelength no longer depends solely upon the ratio of the coefficients of viscosity but also upon the ratio of density of the medium to the applied pressure. *D. R. Bland* (Manchester)

495:

★Iljushyn, A. On stress-small strain relations in the mechanics of continuous media. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 31–34. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi + 362 pp.

Since plastic strain is assumed incompressible, strain can be represented by a vector in a five dimensional space. For an arbitrary material point, the tip of this vector describes a curve as the material is loaded. The author assumes that the stress deviation can be written as a function of the strain and that the material is isotropic. He then applies the Frenet formulas to the strain curve to draw conclusions concerning the general form of the stress-strain relation. *P. G. Hodge* (Chicago, Ill.)

496:

Teodorescu, Petre P. Sur le problème plan de la thermoélasticité. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 471–479. (Romanian. Russian and French summaries)

In the present paper the author studies thermo-elastic stress distribution in the interior of a homogeneous and isotropic body. Two cases are dealt with in particular, viz., (a) the temperature distribution, $T(x, y)$, is known inside the body, and (b) the function $T(x, y)$ satisfies the heat equation. He also indicates some boundary-conditions that should be satisfied by the appropriate stress-functions, as well as some methods of approximate analysis. It should be noted, however, that the author does not present any proofs as to whether these boundary conditions are compatible.

Existence and uniqueness theorems are not stated.

K. Bhagwandin (Oslo)

STRUCTURE OF MATTER

See also 374, 608.

497:

★Houwink, R. Elasticity, plasticity and structure of matter. 2nd ed. Dover Publications, Inc., New York, 1958. xviii + 368 pp. $2.45.

This is an unabridged republication of the recent edition [Harren, Washington, D.C., 1953] of this well-known treatise on experimental rheology. The difference from the 1937 edition is only slight, principally the bringing up to date of the section on molecular structure of rubber and related statistical theories of its elastic properties.

R. Hill (Nottingham)

498:

D'Heedene, R. N. Simultaneous invariance of generalized spherical harmonics under the operations of two rotation groups. Quart. Appl. Math. **16** (1958), 188–192.

A rigid molecule is situated in a lattice. Two systems of coordinates one molecule-fixed (x, y, z) and one lattice-fixed (X, Y, Z) are given. A group of rotations of (X, Y, Z) and of (x, y, z) describing the symmetry of the lattice and the symmetry of the molecule are denoted by G and H respectively. The potential energy of the molecule V as represented as a series of generalized spherical harmonics is required to be invariant under each operation in G or H. A method is found for the evaluation of the coefficients of this series. *F. Oberhettinger* (Madison, Wis.)

499:

Wolff, P. A. Theory of plasma resonance in solids. Phys. Rev. (2) **112** (1958), 66–69.

The paper explains the connection between two formulae for the frequency of modes of plasma resonance in solids. One of these formulae has been given by Dresselhaus, Kip, and Kittel [Phys. Rev. **100** (1955), 618–625] (referred to as DKK), the other can be found for instance in D. Pines's article in *Solid state physics* [Vol. I, ed. by Seitz and Turnbull, Academic Press, New York, 1955; MR **19**, 793; pp. 367–450]. It is found that in a sample whose size is comparable with the Debye length, one indeed finds the various frequencies given by Pines, but for larger samples these frequencies are too closely spaced to be resolvable and, in fact, almost coincide at the frequency given by DKK. The actual calculations are carried out for the geometry of a thin foil. It is also remarked that the actual observation of the split-up DDK level in small samples would be made difficult by the line broadening through surface scattering.

M. J. Moravcsik (Livermore, Calif.)

500:

Rumanova, I. M. Ordinary and band weighted projections of electron density with initial phase. Akad. Nauk SSSR. Kristallografiya **3** (1958), 664–675. (Russian)

Weighted phase projections of the type

$$C_L(x, y, Z) = \int_0^C \rho(x, y, z) \cos \frac{2\pi L}{C} (z-Z)dz,$$

$$S_L(x, y, Z) = \int_0^C \rho(x, y, z) \sin \frac{2\pi L}{C} (z-Z)dz$$

are introduced. These permit the determination of all three atomic coordinates from the structural factor F_{hkL} of the reflections of the L-layer line ($L \neq 0$). They also permit the refinement of the x and y coordinates of atoms which are superimposed in the (xy) projection. (The method is inapplicable only if $z_I/c - z_{II}/c = \frac{1}{2}$ for atoms I and II.) When the crystal has a plane of symmetry perpendicular to the c-axis, the upper limit of integration in the above expressions is replaced by $c/2$, the projections being denoted by $C_L^{[0,\, c/2]}(x, y, z)$, etc. These expressions, however, cannot be used if the c-axis is an improper twofold axis of rotation. Band projections, in which the limits of integration are z_1 and z_2, are introduced for use in such a case. They are evaluated explicitly for $z_1 = c/4$, $z_2 = 3c/4$.

J. E. Rosenthal (Passaic, N.J.)

FLUID MECHANICS, ACOUSTICS

See also 491, 492, 570.

501:

Gold, Richard R.; and v. Krzywoblocki, M. Z. On superposability and self-superposability conditions for hydrodynamic equations based on continuum. I, II. J. Reine Angew. Math. **199** (1958), 139–164; **200** (1958), 140–169.

These papers constitute an extremely full review of what can be said about conditions under which velocity fields of fluids of all kinds may be added up to produce new velocity fields satisfying the equations of motion. The arguments for investigating the matter are unconvincing to this reviewer.

M. J. Lighthill (Manchester)

502:

Vulis, L. A.; and Kaškarov, V. P. Self-modelling flow motions of a fluid. I, II. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. no. 6 (10) (1957), 3–19. (Russian. Kazah summary)

The starting point in Part I is the Navier-Stokes equation of motion for a constant density and viscosity coefficient fluid in spherical coordinates with an axis of symmetry so that the derivatives with respect to one of the coordinates vanish. The authors assume the concept of a separation of variables and represent each of the dependent functions (velocity components and pressure) in terms of products of two separate functions of two coordinates (angle and radius), the dependance on the radius being in form of a power of the radius. By proper choice of the power coefficients, the partial differential equations transform into ordinary differential equations. The involved functions are only those of the angle. These are called by the authors self-modelling flow motions. Obviously, many particular cases of flows discussed by previous authors [like H. B. Squire, Philos. Mag. (7) **43** (1952), 942–945; MR **14**, 1139], employing ordinary differential equations, are contained in the present analysis. The same technique is applied to equation of energy (heat diffusion) and to the set of equations in plane polar coordinates. In Part II this technique is applied to equations of a turbulent motion. In expressing the Reynolds stresses the authors use the concept of turbulent kinematic viscosity and heat conductivity coefficients. The turbulent Prandtl number is assumed to be constant. The following self-modelling flow motion equations are derived: axially symmetric flow with the constant turbulent kinematic viscosity coefficient; heat diffusion in such a flow; curvilinear flow (x, r); two-dimensional plane flow, where the derived equations are similar to those in the boundary layer. Obviously, the technique generalizes the methods used in the boundary layer theory.

M. Z. v. Krzywoblocki (Urbana, Ill.)

503:

Dumitresku, D. Axially symmetrical motion of liquids. Dokl. Akad. Nauk SSSR **123** (1958), 963–966. (Russian)

The author solves the problem of axially symmetrical motion of incompressible liquids by two methods. The starting point is the velocity potential and the stream function equations in cylindrical polar coordinates (ρ=const., axial symmetry). The first method involves Bessel polynomials, whose structure takes into account the boundary conditions on the external radius. The second approach to the solution is by means of the Southwell method. Both techniques are applied to find the streamlines of a flow inside round tubes with variable cross-sections giving very nice results in two cases: a step-wise change in the cross-section and a smooth (a half-sphere) transition between two cross-sections inside the tube.

M. Z. v. Krzywoblocki (Urbana, Ill.)

504:

Godal, Thore. On Beltrami vector fields and flows ($\nabla \times V = \Omega V$). III. Some considerations on the general case. Univ. Bergen Årbok. Naturvit. Rekke **1957**, no. 12, 28 pp. (1958).

[For parts I and II, see O. Bjørgum, same Årbok. **1951**, no. 1 (1952); MR **15**, 569; and Bjørgum and Godal, ibid. **1952**, no. 13 (1953); MR **15**, 570]. Here the author for-

mulates theorems of local existence and uniqueness for steady isotropic flow, generally rotational: 1°, corresponding to prescribed velocity, pressure, and entropy upon a surface element; 2°, within a prescribed stream tube with speed, entropy, and abnormality prescribed upon one cross-section, and with pressure prescribed on a certain curve. He then discusses general solution of the integrability condition curl $(w\times v)=0$. The results are then applied to determination of Beltrami fields $(w\times v=0)$ subject to particular geometric restrictions. Finally, by use of the calculus of variations, the author constructs an analogy between the vector lines of a Beltrami field and the equilibrium positions of heavy strings suspended in a conservative field of force. *C. Truesdell* (Bloomington, Ind.)

505:

Ghildyal, C. D. Steady self-superposable flows of the type curl $q=\lambda q$. Gaṇita 8 (1957), 61–69.

Steady flows in both viscous and non-viscous homogeneous fluids satisfying the Beltrami condition curl $\bar{q}=\lambda\bar{q}$ are discussed. Particular solutions are obtained by assuming velocity components (i.e. the components of $\bar{q}$) and also the proportionality factor λ independent of z.

The Beltrami condition, together with the equation of continuity and Navier's dynamical equations, is used. It is shown that steady flows of the Beltrami type in a viscous homogeneous incompressible fluid are always irrotational when λ is assumed to be a function of two space variables only. However, in the case of non-viscous fluid, two particular solutions are obtained. In case $\lambda=1/r$, where $r^2=x^2+y^2$, explicit expressions for the velocity components u and v are found:

$$u=\frac{Ay}{r^{3/2}}\cos\left(\frac{\sqrt{3}}{2}\log r+\varepsilon+\frac{\pi}{6}\right),$$

$$v=-\frac{Ax}{r^{3/2}}\cos\left(\frac{\sqrt{3}}{2}\log r+\varepsilon+\frac{\pi}{6}\right),$$

A and ε are arbitrary constants. Similarly, in the case $\lambda=r$ explicit expressions for u and v are found in terms of gamma functions and Bessel functions.

J. F. Blackburn (New York, N.Y.)

506:

Ballabh, Ram. On Beltrami flows in the atmosphere. Gaṇita 8 (1957), 41–49.

An investigation is made of the existence of flows such that curl $\bar{q}=\lambda\bar{q}$ in the earth's atmosphere at a height where the temperature may be assumed to be constant. Compressibility of air, variations in gravity and curvature of the earth's surface are neglected. $\bar{q}$ is velocity and λ a proportionality scalar.

Using the equations governing the motion of a viscous heterogeneous incompressible fluid, the condition of incompressibility, the equation of continuity and the Beltrami condition curl $\bar{q}=\lambda\bar{q}$, a set of integral expressions is obtained for the velocity components u, v, and w.

Two cases are then considered. In the barotropic case, the following expressions for the velocity components are obtained:

$$u=Ce^{-B^2t}\sin\omega,\quad v=Ce^{-B^2t}\cos\omega,\quad w=0,$$

where

$$\omega=-\frac{B^2}{A-2\Omega_z}\log\{D-(A-2\Omega_z)z\}+At+A',$$

A, B, C, D and A' are absolute constants and Ω is the earth's angular velocity of rotation. A less restrictive non-barotropic case is then considered. This results in simplified integral expressions for the velocity components.

The author thus shows that Beltrami flows capable of reinforcing themselves can exist in the upper atmosphere. He shows that the paths of particles for the solutions vary from spiral curves to circles.

J. F. Blackburn (New York, N.Y.)

507:

★**Seth, B. R. Non-linear rotational flows.** Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 199–202. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

Some particular solutions of the two dimensional vorticity equation $\nabla^2\psi=f(\psi)$ are obtained when $f(\psi)$ is non-linear. The cases $f(\psi)=(4/n)e^{n\psi}$ and $f(\psi)=4\psi^n$ are discussed in detail and several examples are given.

M. Schechter (New York, N.Y.)

508:

Spence, D. A. Some simple results for two-dimensional jet-flap aerofoils. Aero. Quart. 9 (1958), 395–406.

These results pertain to the limiting case of the same author's equations [Proc. Roy. Soc. London, Ser. A. **238** (1956), 46–48; MR **18**, 529] for zero incidence α and small jet deflection τ, when the jet coefficient C_J is very small. Approximate expressions for lift coefficient C_L and pressure distribution are obtained.

From the form of these results and from the numerical results of the previous paper (for larger values of C_J), new interpolation formulas for $\partial C_L/\partial\alpha$ and $\partial C_L/\partial\tau$, less cumbersome than those previously given, have also been determined and are presented here.

W. R. Sears (Ithaca, N.Y.)

509:

Gheorghiţă, Şt. I. Problèmes d'hydrodynamique rélatifs à l'extérieur d'un cylindre semicirculaire. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 6 (1957), no. 13, 15–25. (Romanian. French and Russian summaries)

The author investigates hydrodynamic forces on semicircular cylinders for a number of different angles of attack. The results, presented in the paper, are not altogether new. However, the conformal mapping method employed by the author yields solutions in terms of elementary functions. The author's results agree with those found by previous investigators.

K. Bhagwandin (Oslo)

510:

★**Kaplan, Paul; and Hu, Pung Nien. The forces acting on slender submerged bodies and body-appendage combinations in oblique waves.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 761–769. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

"The hydrodynamic forces and moments acting on a slender submerged body of revolution moving obliquely to the crests of regular waves are found by the application of slender-body theory. The forces acting on body-appendage combinations are considered and solutions

obtained for certain stern fin configurations." (From authors' summary)

L. E. Payne (Newcastle-upon-Tyne)

511:

Carafoli, E.; et Ionescu, M. Théorie générale de l'aile triangulaire à distribution de pression donnée (problème indirect). Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 267–284. (Romanian. Russian and French summaries)

The problem treated in this paper is the following: given the pressure distribution over a delta wing, with subsonic leading edges, placed in a supersonic free stream, to calculate the resulting vertical velocity component. This is usually called the "indirect" problem. The authors derive their results from the known solution of the "direct" problem; they show that the two problems are very closely related. Two particular examples are included: (i) the case of conical flow, (ii) the case of a linear pressure distribution.

G. N. Lance (Southampton)

512:

Polyahov, N. N. Aerodynamic theory of an oscillating wing of finite span. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 19, 87–97. (Russian. English summary)

Generalizing work of Birnbaum and Kussner for the case of infinite span, the author develops a linearized flow theory for a thin rectangular airfoil of finite span, which oscillates harmonically. The analysis leads to an integral equation, which the author solves in the special case of infinite aspect ratio and circulation depending only on time.

R. Finn (Stanford, Calif.)

513:

Anliker, Max. A numerical method of evaluating the velocity potential and the minimum drag warping of arbitrary supersonic wings. Z. Angew. Math. Phys. **10** (1959), 1–15. (German summary)

Cette étude développe dans le cadre de l'approximation linéaire les principales formules nécessaires à la résolution par machines de certains problèmes relatives aux ailes en écoulement supersonique. Sont établies, d'abord les formules donnant les coefficients d'influence pour le potentiel des vitesses (potentiel du à une incidence unité dans un petit rectangle dont les diagonales sont parallèles aux lignes de Mach), puis la formule donnant la trainée. Une application est faite au problème général de la détermination d'une aile de forme en plan donnée devant réaliser le minimum de trainée sous certaines conditions imposées à l'avance soit d'ordre géométrique, soit d'ordre aérodynamique. Une comparaison avec certains résulats obtenus par voie analytique, justifie pratiquement la validité de la méthode préconisée.

P. Germain (Paris)

514:

Guiraud, Jean-Pierre. Lignes de courant d'un écoulement newtonien. C. R. Acad. Sci. Paris **247** (1958), 736–738.

Étant donné un écoulement hypersonique, tridimensionnel, d'un fluide parfait, sur une surface, l'Auteur montre que dans la couche de choc le réseau des lignes de courant est généralement confondu avec celui des géodésiques.

R. Gerber (Grenoble)

515:

Guiraud, Jean-Pierre. Écoulement newtonien sur une surface. C. R. Acad. Sci. Paris **247** (1958), 775–778.

Complétant la note analysée ci-dessus, l'auteur établit les formules résolutives qui achèvent de déterminer complètement l'écoulement dans la couche de choc d'épaisseur infinitésimale.

R. Gerber (Grenoble)

516:

Takano, Kenzo. Effet de passage d'un houle linéaire plane sur un seuil. C. R. Acad. Sci. Paris **248** (1959), 1768–1771.

The problem of reflection of progressing waves over a rectangular step in the bed of a stream is treated by means of Fourier series developments. This leads to an infinite set of linear equations for the coefficients of the series; it is stated that approximate numerical solutions have been obtained by solving a subset of these equations.

J. J. Stoker, Jr. (New York, N.Y.)

517:

Phillips, O. M. The scattering of gravity waves by turbulence. J. Fluid Mech. **5** (1959), 177–192.

If gravitational surface waves pass through a liquid in turbulent motion, they may be scattered and dissipated by action of the turbulence. The problem of the scattering is attacked by separating the velocity field into a surface-induced component characteristic of the wave motion and a vorticity-induced component characteristic of the turbulence. The theory is developed in terms of the Navier-Stokes equations, using Fourier-Stieltjes transforms to represent the variations of the acceleration potential and the vorticity, and assuming the wave amplitudes to be small. The scattering is then described by assuming negligible change of intensity in the primary wave, and an expression is derived for the angular distribution of scattering in terms of a scattering function dependent on the primary wave-number and the spectral distribution of vorticity in the liquid. This scattering function, $\Psi(K)$, has the same dimensions as the two-dimensional vorticity spectrum and it is suggested that it will be directly proportional to this spectrum. On this basis, the directional distribution of the scattered waves is found to be

$$\begin{aligned} s(\theta, k) &= \frac{\pi k^2}{(gk)^{1/2}}\Psi(K) \\ &= 2^{-2/3} B\pi g^{-1/2} \varepsilon^{2/3} k^{5/6} (\sin \tfrac{1}{2}\theta)^{-2/3} \end{aligned}$$

for not too small angles of scattering.

A. A. Townsend (Cambridge, England)

518:

Miles, J. W. On the sloshing of liquid in a flexible tank. J. Appl. Mech. **25** (1958), 277–283.

The author studies coupled bending and free surface "sloshing" of a liquid in a flexible tank. Using Lagrangians, he calculates the effect on the frequencies of cantilever and free-free bending oscillations of a free surface, for a light cylindrical tank nearly full of liquid.

G. Birkhoff (Cambridge, Mass.)

519:

Sarpkaya, Turgut. Oblique impact of a bounded stream on a plane lamina. J. Franklin Inst. **267** (1959), 229–242.

"In this paper, Helmholtz's free streamline theory is

used to determine the contraction coefficients of two jets formed by a finite, two-dimensional stream impinging obliquely upon a plane lamina placed between two infinite parallel planes. The agreement between the results obtained from experiments and those calculated from theory is excellent. Thus, the theory is expected to be useful as a design tool for such boundary configurations as those represented by butterfly valves." (Author's abstract)

R. M. Morris (Cardiff)

520:

★Kapur, J. N. Steady compressible viscous flow through a circular pipe. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 243–250. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi + 362 pp.

In this paper, the author considers the steady compressible viscous non-adiabatic flow through a circular pipe of of a polytropic gas. For a specific class of flows, in which the velocity depends exponentially on the pipe length, it is shown that solutions of the flow equations will not exist unless either the coefficient of viscosity is variable or Stokes' condition is modified. The method of solution is by separation of variables. The author states that this result indicates that a solution of this problem by Ray [Proc. Nat. Inst. Sci. India. Part A **22** (1956), 408–418] is inconsistent with the flow equations. (The reviewer was unable to obtain Ray's paper and hence could not verify the author's remarks.)

N. Coburn (Ann Arbor, Mich.)

521:

Gheorghiţă, Şt. I. Le mouvement lent stationnaire des fluides visqueux dans la présence de quelques enveloppes poreuses nonhomogènes. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 16, 15–23. (Romanian. French and Russian summaries)

In the present paper the author presents a general method of solution for viscous fluid flow in non-homogeneous media. Slow motion, bounded by symmetric enveloping regions, is considered with a particular type of filtration coefficient. A detailed analysis is presented for the particular cases of spherical and cylindrical regions. It should be noted, however, that the filtration-coefficient employed by the author is by no means of a general character, although it seems reasonable from the point of view of the typical problems studied.

K. Bhagwandin (Oslo)

522:

Tirskiĭ, G. A. Nonstationary flow with heat transfer in a viscous, incompressible fluid between two revolving discs accompanied by fluid influx. Soviet Physics. Dokl. **119** (**3**) (1958), 237–239 (226–228 Dokl. Akad. Nauk SSSR).

The paper contains a very terse analysis of the problem of flow and heat transfer existing between two porous disks both of which move with time-dependent angular velocities. The solution is sketched and is not given explicitly, the final results having been given in terms of two functions and a set of eigenfunctions which satisfy appropriate differential equations. The latter must be solved numerically for each particular case.

J. Kestin (London)

523:

Dorfman, L. A. Thermal boundary layer on a rotating disc. Soviet Physics. Dokl. **119** (**3**) (1958), 248–251 (1110–1112 Dokl. Akad. Nauk SSSR).

Comparing the averaged Navier-Stokes equation for the tangential velocity in cylindrical co-ordinates with the corresponding energy equation (in which, as is usual, the dissipation function has been omitted), the author notes that they are analogous for $Pr=1$ if the temperature at the wall is $\sim r^2$. The analogous quantities are: the tangential velocity and the temperature function $\theta = T/r^2$. This leads to a form of the Reynolds analogy and a relation between the shearing stress and the heat flux. The problem is extended to a disk in an axial stream. The case of arbitrary temperature distribution is solved by integration.

The paper must have been badly written, but it is nearly incomprehensible in its translated form.

J. Kestin (London)

524:

★Görtler, H. (Redakteur) Grenzschichtforschung. Boundary layer research. Symposium Freiburg/Br. 26. bis 29, August 1957. Internationale Union für Theoretische und Angewandte Mechanik. Springer-Verlag, Berlin–Göttingen–Heidelberg, 1958. xii + 411 pp. DM 67.50.

This extremely interesting volume contains 32 papers and brief discussions on them presented at a closed symposium on boundary layer research organized at Freiburg in Germany under the auspices of IUTAM. Realizing that fruitful discussion is possible only with a small circle of participants, the organizers limited attendance to some 84 workers drawn from 17 countries. These included many, but not all of the leading names one might think of.

Probably the most baffling problem in boundary layer research is the study of the mechanism of transition. Turbulent flow is essentially non-steady and the velocities in the whole field oscillate in a random manner. On the other hand, any attempt to analyze the process of transition from the truly steady laminar regime to the quasi-steady turbulent regime must begin with the consideration of some regular disturbances for mathematical reasons. It is therefore not surprising that theories which are unable to deal with random functions succeed at best in giving only very crude agreement with experiment but fail to reproduce the intricate details of the process of transition, notably the appearance of turbulent "spots". Just what is involved in the simplest case of a flat plate in subsonic flow is beautifully discussed by G. B. Schubauer in his paper on the "Mechanism of transition at subsonic speeds". At one time it was thought that the Tollmien-Schlichting theory of stability which considers the growth and decay of two-dimensional disturbances explained transition in a satisfactory manner. Recent experimental work, of which that described by Schubauer constitutes a brilliant example, shows a baffling combination of agreement and discrepancy with the theory. In particular, these experiments show conclusively that the full story is not likely to be unfolded in other than a three-dimensional context. H. Görtler and H. Witting contribute a "Theory of secondary instability of laminar boundary layers" which must command the attention of every student of the subject. C. C. Lin discusses, necessarily inconclusively, some aspects, notably the effect of finite amplitudes and non-linear effects in the critical layer. J. A. Zaat contributes the results of refined calculations on curves of neutral stability,

and L. N. Howard examines the stability of the two-dimensional jet. E. A. Eichelbrenner and R. Michel provide a study of transition on an ellipsoid, i.e. in a case when even the free stream is essentially three-dimensional.

The volume under review contains much else besides which is of interest, including problems involving heat transfer, rarefied gases, compressible flows, shock wave attenuation, suction, non-homogeneous (two-component) flows and separation in two- and three-dimensional flows. To discuss them all in detail would exceed the scope of a review in this journal. It is interesting, however, to mention that Prof. M. Reiner (with B. Popper) still believes (after G. I. Taylor's study) in the existence of cross-stresses in air.

Contrary to expectation, the contents of the contributions in the discussion are somewhat disappointing.

The reviewer is certain that the volume under review will provide much stimulation to workers interested in all aspects of boundary layer research for some time to come.

J. Kestin (London)

525:

Liu, V. C. On the separation of gas mixtures by suction of the thermal-diffusion boundary layer. Quart. J. Mech. Appl. Math. **12** (1959), 1–13.

The author proposes a new method for the separation of gases of different molecular weights which makes use of the phenomenon of thermal diffusion. The mixture is passed over a heated porous plate and separation occurs in the thin boundary layer formed on it owing to the large transverse temperature gradient. The enriched gas mixture is sucked away through the hot porous wall. The rate of separation is analysed mathematically from first principles under plausible physical assumptions. The formulation of the expression for the diffusive flux is based on the Chapman-Enskog theory of diffusion. The separation rate is calculated for the asymptotic suction profile, but, in addition, the time required to attain the steady state is estimated with the aid of a simplified form of the governing time-dependent differential equation. An estimate of the optimum suction rate for stability is also given, because one of the main advantages claimed for the method, as compared with the Clusius and Dickel method, is its ability to operate at higher Reynolds numbers in the laminar regime.

J. Kestin (London)

526:

Wu, Ching-Sheng. The three-dimensional incompressible laminar boundary layer on a spinning cone. Appl. Sci. Res. A **8** (1959), 140–146.

By means of suitable transformations, the equations of the problem are reduced to ordinary differential equations, which are solved numerically. Some of the numerical results are shown to follow directly from the solution for the infinite disk obtained by v. Karman [Z. Angew. Math. Mech. **1** (1921), 235–252] and Cochran [Proc. Cambridge Philos. Soc. **30** (1934), 365–375].

D. W. Dunn (Ottawa, Ont.)

527:

Clarke, John F. Energy transfer through a dissociated diatomic gas in Couette flow. J. Fluid Mech. **4** (1958), 441–465.

The influence of reaction rates, both within the fluid and at a wall, is studied for Couette flow of an idealized dissociating gas, with emphasis on heat transfer. The problem is regarded throughout as an illustrative model for the more difficult one of the boundary layer. The two extremes of equilibrium and frozen flow are investigated in detail: in the latter case the catalytic effect of the wall is found to be important. Numerical results are given for oxygen with a cool fixed wall.

M. D. Van Dyke (Paris)

528:

Carrier, G. F.; and Chang, C. T. On an initial value problem concerning Taylor instability of incompressible fluids. Quart. Appl. Math. **16** (1958), 436–439.

In their study of the effect of surface tension and viscosity on Taylor instability, Bellman and Pennington [same Quart. **12** (1954), 151–162; MR **16**, 83] encountered a difficulty in treating the problem when the motion is started from rest, a difficulty which they attributed to their linearized analysis. In order to resolve this difficulty, the authors consider an initial value problem in which the motion is started from rest with an initial periodic displacement of the surface. They show that the solution can be expressed in terms of the roots of a certain characteristic quartic, from which it then follows that the rate of growth of the surface displacement is not of the usual simple exponential type. [Because of the form in which the spatial dependence of the solution is taken, it is necessary to reject all roots with negative real parts, a condition that appears to have been overlooked.]

W. H. Reid (Providence, R.I.)

529:

Howard, Louis N. Hydrodynamic stability of a jet. J. Math. Phys. **37** (1959), 283–298.

The author considers the stability of the two-dimensional laminar jet with respect to the antisymmetric disturbance (for which the eigenfunction $\varphi(y)$ is an even function of y). The problem is converted into an integral equation eigenvalue problem, which is solved by means of expansions of the eigenvalue C and eigenfunction $\varphi(y)$ in powers of the wave-number α, the Reynolds number R being regarded as fixed. The method, which is restricted to the lower branch of the neutral curve, gives the asymptotic result $\alpha \sim 0.954R^{-2}$ for $R \to \infty$ and an approximate value of about 4 for the minimum critical Reynolds number. [These results are similar to those obtained by Tatsumi and Kakutani using a different method: J. Fluid Mech. **4** (1958), 261–275; MR **20** #567].

D. W. Dunn (Ottawa, Ont.)

530:

Moser, J. Remarks on the preceding paper of Louis N. Howard. J. Math. Phys. **37** (1959), 299–304.

The conclusions obtained by Howard (see previous review) depend on the assumption that $\varphi(y)$ and C possess asymptotic (possibly convergent) expansions in powers of α. In the present paper, it is proved that these expansions are convergent for small enough values of α.

D. W. Dunn (Ottawa, Ont.)

531:

Szablewski, W. Zur Theorie der freien Turbulenz von Gasen stark veränderlicher Dichte. Rev. Math. Pures Appl. **1** (1956), no. 3, 181–188.

This is a brief version of the paper reviewed earlier [Ing.-Arch. **25** (1957), 10–25; MR **19**, 89].

Y. H. Kuo (Peking)

532:

Özoklav, Hasan. Flow of a compressible fluid in a hyperbolic channel. J. Math. Mech. 8 (1959), 27–45.

A particular two-dimensional rotational flow of a "gas" without viscosity or heat conduction, whose equation of state is determined as part of the calculation, is derived by assuming that the streamlines are confocal hyperbolae.

M. J. Lighthill (Manchester)

533:

Moriguchi, Haruo. A new approach to the theory of slender body in compressible flow. J. Phys. Soc. Japan **13** (1958), 1384–1390.

Suppose that a solution of the equations for steady, plane, irrotational flow, uniform at infinity, of a perfect compressible fluid past an obstacle is known. Then for flow at another stream speed the differential equations can be written with the original stream function and velocity potential as independent variables. If the difference of stream speed is small, these equations can be linearized approximately. One form of the resulting equations was used by Bers [Comm. Pure and Appl. Math. **7** (1954), 79–104; MR **16**, 84]. If, in addition, the body is slender, an approximate solution can be deduced which explicitly relates the stream functions of the original and perturbed flow patterns. This amounts to the transformation formula given by Imai [J. Phys. Soc. Japan **3** (1948), 352–356; J. Math. Phys. **28** (1949), 173–182; MR **11**, 222; **13**, 84] and by Imai and Hasimoto [J. Math. Phys. **28** (1950), 205–214; MR **11**, 553]. However, the difference in formulation is such that here the boundary conditions are satisfied exactly and a successive-approximation procedure is suggested. The analogous axisymmetric case is also treated; this is compared with the results (analogous to Imai's) obtained by Kusukawa [J. Phys. Soc. Japan **9** (1954), 605–610; **10** (1955), 1093–1101; MR **16**, 301; **17**, 798]. A slide rule for the correction of surface flow speed for changes of flow Mach number is worked out.

W. R. Sears (Ithaca, N.Y.)

534:

Pien, Yen-kwei. The solution of Tricomi's equation for a transonic jet. Sci. Sinica **7** (1958), 946–963.

Le problème du jet transsonique consiste, du point de vue mathématique, en la recherche d'une solution de l'équation de Tricomi satisfaisant aux conditions aux limites d'un problème de Tricomi particulier (le domaine dans le demi plan elliptique est une demie bande). L'auteur résoud explicitement le problème en suivant la méthode générale de Tricomi : il forme les équations intégrales liant les valeurs de la fonction et de sa dérivée normale sur l'axe parabolique, élimine une de ces fonctions et résoud l'équation intégrale singulière ainsi obtenue par une méthode généralisant celle proposée par Carleman [Ark. Mat. Astron. Fys. **16** (1922), no. 26].

P. Germain (Paris)

535:

Landahl, Mårten T. Theoretical studies of unsteady transonic flow. I. Linearization of the equations of motion. Flygtekn. Försöksanstalt. Rep. no. 77 (1958), 18 pp.

L'équation transsonique non linéaire pour le potentiel des vitesses de perturbation d'un mouvement non stationnaire est rétablie. Soit $\varphi(x, y, z, t)$ le potentiel des vitesses de perturbation ; lorsque l'on se limite à l'étude de problèmes de flottement, c'est à dire à l'étude de la stabilité d'un écoulement stationnaire, il est toujours possible de supposer φ de la forme

$$\varphi = \varphi_1(x, y, z) + \varphi_2(x, y, z, t)$$

avec $\varphi_2 \ll \varphi_1$. En faisant usage des lois de similitude transsonique relatives à ϕ_1 on montre que l'équation vérifiée par ϕ_2 est :

$$(1) \qquad \varphi_{2yy} + \varphi_{2zz} - 2M^2\varphi_{2xt} - M^2\varphi_{2tt} = 0,$$

sous réserve que soit vérifiée la condition $k \gg |I-M|$, où k désigne la fréquence réduite, et M le nombre de Mach de l'écoulement général, et sous réserve également que soient vérifiées les conditions de validité de l'équation transsonique non linéaire pour φ_1.

Le reste du mémoire est consacré à une discussion des propriétés comparées de l'équation (1) et de l'équation acoustique

$$(2) \quad (1-M^2)\varphi_{2xx} + \varphi_{2yy} + \varphi_{2zz} - 2M^2\varphi_{2xt} - M^2\varphi_{2tt} = 0.$$

La discussion montre que l'équation acoustique (2) conduit à des solutions physiquement inadmissibles dans le domaine transsonique. Ce fait provient du comportement du système d'ondes rétrogrades, à propagation lente, ondes qui remontent, avec la célérité locale du son, l'écoulement local, mais sont entrainées en aval par ce dernier. L'auteur montre que la propagation de ces ondes rétrogrades est fortement influencée par le détail de l'écoulement stationnaire de base caractérisé par φ_1 ; ce point est complètement ignoré par la théorie acoustique. En règle générale, lorsque la condition $k \gg |I-M|$ est vérifiée, c'est à dire lorsque la longueur d'onde des ondes rétrogrades est très faible, l'on obtient un résultat correct en admettant que ces ondes restent dans des plans $x = C^{te}$, conformément à (2). Toutefois, cette règle souffre des exceptions, et l'auteur met en garde contre l'utilisation de l'équation (2) pour prédire le phénomène connu sous le nom de "Buzz d'aileron". Dans ce cas, les ondes rétrogrades prenant naissance sur l'aileron peuvent remonter le long de ce dernier et aller se réfléchir sur l'onde de choc. Quelques autres conclusions intéressantes, physiquement interprétables sont mises en évidence par l'étude fine locale de ce phénomène de propagation ; telle est, par exemple, l'instabilité des écoulements décélérés au passage de la vitesse du son.

Ce rapport est le premier de toute une série consacrée aux écoulements transsoniques non stationnaires (voir les deux analyses suivantes).

J. P. Guiraud (Paris)

536:

Landahl, Mårten T. Theoretical studies of unsteady transonic flow. II. The oscillating semi-infinite rectangular wing. Flygtekn. Försöksanstalt. Rep. no. 78 (1958), 20 pp.

Ce rapport fait suite au rapport précédemment analysé. En effectuant une transformation de Fourier sur la variable x, l'équation transsonique non stationnaire est ramenée à une équation de Helmoltz. L'on est ainsi conduit à un problème aux limites analogue à celui qui est posé par la diffraction des ondes acoustiques par un demi-plan. Après retour aux variables physiques la solution est exprimée, pour une déformation arbitraire de l'aile, à l'aide d'une intégrale triple. Le cas où la déformation est indépendante de l'envergure est plus spécialement examiné. Finalement les efforts généralisés sont calculés,

uniquement pour des mouvements d'ensemble, battement et torsion.

Soit k la fréquence réduite et σ le rapport demi-envergure sur corde pour une aile rectangulaire. Lorsque $k\sigma^2 \gg I$ les résultats précédents permettent d'obtenir une approximation convenable en négligeant l'interaction mutuelle des deux bords. Dans le rapport suivant une théorie est développée pour l'aile rectangulaire lorsque $k\sigma^2 \ll I$. Il se trouve que les résultats des deux théories se recoupent en gros, ceci au voisinage de $k\sigma^2 = 0{,}5$.

J. P. Guiraud (Paris)

537:

Landahl, Mårten T. Theoretical studies of unsteady transonic flow. III. The oscillating low aspect ratio rectangular wing. Flygtekn. Försöksanstalt. Rep. no. 79 (1958), 15 pp.

Ce rapport fait suite aux deux rapports précédemment analysés. Soit k la fréquence réduite et σ le rapport de la demienvergure à la corde, la théorie développée suppose $k\sigma^2 \ll 1$. Après transformation de Fourier sur la variable x, l'auteur utilise une technique utilisée par Adams et Sears dans la théorie des corps élancés. Le résultat n'est pourtant pas exactement le même que celui que donnerait l'application directe du procédé d'Adams et Sears car l'aile ne présente pas de pointe à l'amont; il y intervient quelques termes complémentaires. Lorsque la fréquence réduite est très faible, l'on retrouve les résultats de la théorie conventionnelle des corps élancés. Les résultats numériques montrent que pour des ailes dont l'allongement est raisonnable, l'écart à la théorie conventionnelle des corps élançés est important aux fréquences réduites intervenant dans les problèmes de flottement. *J. P. Guiraud* (Paris)

538:

Germain, Paul. Écoulements transsoniques au voisinage d'un point de rencontre d'une onde de choc avec une ligne sonique. C. R. Acad. Sci. Paris **247** (1958), 2290–2292.

Let $\phi(x, r)$ be the perturbation-velocity potential at a point with rectangular coordinates x, r in approximately sonic flow parallel to the x-axis. Seek solutions of $\phi_{rr} - (\gamma+1)\phi_x\phi_{xx} = 0$ of the form $\phi = r^{3n-2}f(\zeta)$, where $(\gamma+1)^{1/3}r^n\zeta = x$, and f satisfies a second order ordinary differential equation. For $s = \zeta^{-3}f$ and $t = \zeta^{-2}f'$ then $s(t)$ satisfies a first order equation (incorrectly printed in the article). The author wishes to employ an arc of such a solution to construct a flow field near the intersection of a shock and a sonic line. A shock in the xr-plane is a curve $\zeta = \text{const.}$ on which s is continuous but t is discontinuous, the mean of t being n^2. By an analysis of the behavior in the large of the integral curves in the st-plane, combined with a Legendre transformation $\chi = xu + rv - \phi$ into the uv-plane, the author determines some necessary conditions for the desired type of flow, in particular $7/5 < n < 5/3$. To complete the determination of a solution the value of a certain parameter must be selected. A series of numerical calculation for $n = 3/2$ and various values of the parameter have all yielded supersonic flow behind the shock.

J. H. Giese (Aberdeen, Md.)

539:

Poluboyarinov, A. K. Solution of the linearized equation of axi-symmetric supersonic gas flow. Vestnik Leningrad. Univ. **12** (1957), no. 13, 102–112. (Russian. English summary)

The determination of an axisymmetric perturbed three-dimensional source flow can be reduced to solving an equation of the form (i)

$$\frac{\partial^2 U}{\partial\xi\partial\eta} + [a(\varphi) - b(\nu)]\frac{\partial U}{\partial\xi} + [a(\varphi) + b(\nu)]\frac{\partial U}{\partial\eta} = 0,$$

where $\varphi = \xi + \eta$, $\nu = \xi - \eta$, (ii) $2a = \cot\varphi$, and (iii) $2b = g(\nu)$. Here φ is an angle measured from the axis of symmetry, and $g(\nu)$ is known in terms of functions that characterize source flow. On the other hand, if (iv) $h = da/d\varphi + a^2 - db/d\nu - b^2$ were to satisfy (v) $\partial^2 \ln h/\partial\xi\partial\eta = h$, then by Laplace's cascade method one could find the general solution of (i) in a form involving an arbitrary function of ξ and another of η. Equations (iv) and (v) have a five-parameter family of solutions a, b which can be described by means of elementary functions or as solutions of first order ordinary differential equations. The author chooses parameters which for $10° \leqq \varphi \leqq 25°$ and $5° \leqq \nu \leqq 25°$ yield a and b that approximate very well the correct functions (ii) and (iii). He also shows how to determine the arbitrary functions in the general solution U of the approximate equation (i) to satisfy the characteristic and non-characteristic initial value problems and a standard mixed characteristic non-characteristic problem related to flow about a given body of revolution.

J. H. Giese (Aberdeen, Md.)

540:

★**Hayes, Wallace D.; and Probstein, Ronald F. Hypersonic flow theory.** Applied Mathematics and Mechanics, Vol. 5. Academic Press, New York–London, 1959. xv+464 pp. $11.50.

The term hypersonic is commonly used to describe the phenomena encountered when a body moves through a gas at a speed so high that "the Mach number is large". More precisely, the gas is meant to have definite density, but a temperature approaching the absolute zero. That such a limit should be of pressing interest might seem improbable, but the word "large" is notoriously elastic. There is little doubt that a substantial number of mathematicians will find themselves obliged to face the problems arising in this field, and a discussion of this very first book on the subject from their point of view is therefore important.

The book is more than just a first, it is undoubtedly a milestone, but not all the inscriptions on it are easy to decipher. Not the least difficulty for the mathematician is that the authors are theoretical physicists and offer not an introduction to the subject, but a mixture of monograph and research memoir addressed largely to scientists already active in the field, especially physicists and engineers. The newcomer will find as much of an introduction as is offered by turning from chapter 1 straight to section 10.1, where the range of the field is surveyed. Much of it is still uncharted, and the book is concerned almost entirely with the end at which the classical approach is valid, where shocks and boundary layers are taken to be very thin and in the steady flow between them viscous effects are neglected.

This orthodox problem, however, is to be solved for bodies with a blunted nose, and much of the outstanding merit of the book lies in the contributions made to the understanding of the flow near the nose. To understand these contributions, in turn, the inexpert reader may find

it easiest to read backwards, from chapter 6, which summarises the numerical solutions known to date. Some triumphs of analysis were needed to obtain the more trustworthy ones, but it remains uncertain just how much general information they furnish. The most obvious fact emerging from them, as from elementary considerations, is that the region of inviscid flow sandwiched between bow shock and body boundary layer is itself a rather thin layer (referred to unfortunately as "shock layer"). It is natural, then, to turn to the approximations made on this basis (chap. 5), but the progress achieved turns out to be disappointing.

The need for radical approximations is now obvious. That the inviscid filling of the sandwich must be thin near the nose follows simply from the fact that the shock raises the density by a rather large factor. Since it also lowers the Mach number drastically, the density can vary by only a few per cent between the shock and the stagnation point, and simple, accurate approximations (chap. 4) are obtained from the assumption that the fluid is incompressible! Unfortunately, this applies only to the immediate neighborhood of the stagnation point. A little further away, the idea of the large density ratio across the shock leads to a different first approximation, and it is fortunate that the book has nudged Hayes to publish his theory of the relation between shock and body shape in chapter 5. The problem is one of transonic flow, but of the type associated with acceleration, and Hayes' idea of representing the transition by the singular point of an ordinary differential equation should be valuable in further work, for which there is much scope.

Faced with so many riddles, the reader will be ready to appreciate why the monograph had to be written on a deductive basis, and why Hayes was led to concentrate first of all on the very extreme limit of a layer between shock and body of strictly zero thickness. This "Newtonian" limit of infinite density ratio leads to the novel and mathematically beautiful theory of flow, not past, but actually in a surface. The known part of this theory is presented briefly, but it is not possible to go far without encountering the most fantastic anomalies, and most of chapter 3 is a research memoir in which Hayes attempts to chart the landmarks of a surrealist world. The reader will wish that Hayes had given proofs for those of his conclusions for which he has them; but in any case, the difficulty of connecting the abstrusities clearly with any known features of less surrealistic approximations leaves little alternative to the theoretical physicist's approach of outlining some striking properties of his models and leaving it to others to make sure whether they represent an image of reality.

Away from the nose of the body, the sandwich is thin in the different sense that the shock, even though strong, forms a slender surface confining the inviscid layer to a region of small angular extent. An earlier survey of the first approximation for this part of the field was given by Van Dyke. In devoting chapter 2 to the same subject, Hayes concentrates on producing a companion piece that completes a clear and definitive picture.

Other approximations for the inviscid layer away from the nose are treated in chapter 7. It forms part of Probstein's half of the book, in which proofs are still rare, but on the whole, pains are taken to explain clearly and fully all the arguments supporting the conclusions presented, much to the reader's relief after the arduous task of puzzling out the depths of Hayes' thought. The discussion of inviscid flows is completed in the same chapter by a formulation of the method of characteristics for a general gas.

The insistence on the general gas is, in fact, one of the pervading features of the book. The need for such a point of view emerges occasionally during the first half, but it becomes overwhelming as soon as the discussion turns to viscous effects, of which the outstanding one is the production of high temperatures or high heat transfer rates by dissipation. The discussion of boundary layers with the secondary dissociation and diffusion effects thereby brought into play occupies chapter 8. Unfortunately, only few quantitative results on the influence of diffusive heat transfer have been obtained to date, and the discussion of turbulent boundary layers must necessarily be sketchy at the present time. Another secondary effect is the thickening of boundary layers, which leads to an interdependence between boundary layer and inviscid flow outside; cases for which the concept of displacement thickness remains valid are treated in chapter 9. While they may not turn up any exciting mathematics, these two chapters constitute an excellent survey of the work on boundary layers with hypersonic complications and add notably to the value of the book.

In the final chapter, the authors turn their backs on the conventional approach with its distinct shocks, boundary layers and inviscid flow regions. The estimates of Probstein and Adams, which have set the stage for much new work in continuum gas dynamics, are given. But only a treatment of the stagnation region without distinguishable inviscid layer has appeared in time for the book. It ends with a survey of the extreme range of flow conditions in which molecular encounters are rare.

There is not room here for an enumeration of the many original and previously unpublished contributions of both authors. Moreover, there are not many pages on which the work of others is reported without illumination by a new point of view. The mathematician may be tempted to deplore the inclusion of material that may prove controversial. Not a little confusion must be anticipated from some authors' basing their work blithely on conclusions here drawn which are at the same time attacked by other authors. But such has always been the progress of science. A book on hypersonic flow is needed by not a few mathematicians and needed especially before the subject has been clarified sufficiently for a mathematical text. They will find reason to be grateful that the authors have brought so much good judgment to so difficult a task. No other book on hypersonic flow is likely to supersede this one for quite a few years.

R. E. Meyer (Providence, R.I.)

541:

Hammitt, Andrew G. The hypersonic viscous effect on a flat plate with finite leading edge. J. Fluid Mech. 5 (1959), 242–256.

Unless the leading edge is extremely sharp, the inviscid pressures due to bluntness dominate those induced by the boundary layer; no region of strong viscous interaction exists. For this situation (which prevails when the leading-edge Reynolds number exceeds about $M^3/10$) the author calculates the surface pressures induced by viscosity as a perturbation of the inviscid flow field. The inviscid pressure distribution at the edge of the boundary layer is taken from experiments at high leading-edge Reynolds numbers rather

than from blast-wave theory (though the latter actually appears to be more accurate downstream of 5 leading-edge thicknesses). The boundary layer equations are solved by a momentum integral method, the displacement thickness is assumed to represent an equivalent solid body, and Ackeret's linearized theory is used to calculate the pressure increment. Thus the increased surface pressures measured at lower Reynolds number are satisfactorily accounted for.
M. D. Van Dyke (Paris)

542:
Froese, Charlotte. On the calculation of the velocity of sound in sea water. Canad. J. Phys. 37 (1959), 775–779.

543:
Lapin, A. D. Scattering of sound waves in irregular waveguides. Soviet Physics. Dokl. **118** (3) (1958), 65–68 (55–58 Dokl. Akad. Nauk SSSR).

Let a waveguide be defined in the xz-plane by the relations $-\infty<x<\infty$, $0\leqq z\leqq h$. The right half of the waveguide ($x>0$) is filled with an inhomogeneous medium with refractive index $1+\mu(x,z)$, $|\mu|\ll 1$, μ regarded as a random variable. A wave with wave number k is incident from left to right. Under the requirement that the field satisfy suitable boundary conditions on the walls of the waveguide, the field potential may be approximated by an expression due to S. M. Rytov [Izv. Akad. Nauk SSSR. Ser. Fiz. **2** (1937), 223–259], provided the gradient of the component scattered by the inhomogeneities is in magnitude of the order of $k|\mu|$. When the inhomogeneities are large, the component of the field scattered to the left is large relative to the incident field and the component scattered to the right is negligible. Representing the total scattered field as a sum of normal modes each approximated as above, the author calculates the mean-square amplitude for the r-th mode.

A second part of the paper deals with roughness in the upper wall of the waveguide by assuming that $z=h+\zeta(x)$, where ζ is a suitably restricted random variable. The possibility of making the same calculations, based on the Rytov approximation, is indicated.
R. N. Goss (San Diego, Calif.)

544:
Samuels, J. Clifton. On propagation of waves in slightly rough ducts. J. Acoust. Soc. Amer. **31** (1959), 319–325.

Assuming duct walls at $x=\pm(a/2)+\varepsilon N(z)$, the author derives formal expressions for the perturbed (relative to the plane-wall case) guided-wave modes through terms of order ε^2 and gives more detailed calculations for the term of order ε and some specific $N(z)$.
J. W. Miles (Los Angeles, Calif.)

545:
Ong, R. S.; and Nicholls, J. A. On the flow of a hydromagnetic fluid near an oscillating flat plate. J. Aero/Space Sci. **26** (1959), 313–314.

This note considers the *MHD* flow near an infinite plate which oscillates parallel to itself in a perpendicular magnetic field. This work is an extension of the problem analysed by Rossow [NACA Tech. Note 3971 (1957)]. As in the latter paper, the magnetic field is assumed constant and the manner in which currents are to be closed, etc. is left unresolved.
H. Greenspan (Cambridge, Mass.)

546:
Rott, Nicholas. A simple construction for the determination of the magnetohydrodynamic wave speed in a compressible conductor. J. Aero/Space Sci. **26** (1959), 249–250.

Étant donné un fluide compressible doué d'une conductivité électrique infinie, l'auteur indique une construction graphique qui permet d'obtenir, en chaque point dans chaque direction, le carré des vitesses de propagation du son.
H. Cabannes (Marseille)

547:
Krook, Max. Structure of shock fronts in ionized gases. Ann. Physics **6** (1959), 188–207.

L'auteur étudie la structure des ondes de choc dans un gaz ionisé par les méthodes de la théorie cinétique. A l'infini amont et à l'infini aval, la distribution des vitesses est supposée Maxwellienne, les deux états étant liés par les équations du choc. Les valeurs intermédiaires de la fonction de distribution sont obtenues en résolvant l'équation de Boltzmann par la méthode de Mott-Smith [Phys. Rev. (2) **82** (1951), 885–892]. La solution est approchée, les quatre premières approximations sont comparées.
H. Cabannes (Marseille)

548:
Patrick, R. M.; and Brogan, T. R. One-dimensional flow of an ionized gas through a magnetic field. J. Fluid Mech. **5** (1959), 289–309. (4 plates)

Les auteurs ont étudié expérimentalement l'influence d'un champ magnétique sur l'écoulement rectiligne d'un gaz ionisé. Ils ont mis en évidence la diminution de la conductivité électrique scalaire (partie sphérique du tenseur) lorsque le champ magnétique augmente, l'existence de l'effet Hall et l'apparition d'un choc sous l'effet du champ magnétique.
H. Cabannes (Marseille)

549:
Golitsyn, G. S. Unidimensional motion in magnetohydrodynamics. Soviet Physics. JETP **35** (8) (1959), 538–541 (776–781 Ž. Eksper. Teoret. Fiz.).

The paper concerns the motion of a perfectly conducting gas in a magnetic field. The Riemann invariants for a number of gases are computed. Some non-stationary problems are solved including the problem of a piston entering a tube containing a perfectly conducting gas, and the reflection of a shock wave from a rigid wall. The author also introduces an approximation for the state law in order to obtain approximate general solutions.
H. Greenspan (Cambridge, Mass.)

550:
Kraus, Lester; and Watson, Kenneth M. Plasma motions induced by satellites in the ionosphere. Phys. Fluids **1** (1958), 480–488.

This is a preliminary investigation of the phenomena associated with the supersonic motion of a charged body in an ionized gas.

When the ion and electron mean free paths are small compared with the distance in which the flow potential changes by an appreciable fraction of itself, there is local thermodynamic equilibrium and fluid behaviour obtains. The linearized equations of motion and the electrostatic equations are investigated by Fourier methods. Following the method of Bohm and Pines [Phys. Rev. (2) **85** (1952),

338–353] an expression is obtained for the electrohydrodynamic drag on an object whose dimensions are small compared with the Debye length.

When the gas density is too low for thermodynamic equilibrium to become established in the wake, recourse must be had to the Boltzmann equations. These are again linearized and investigated by Fourier methods, the collisional effect being treated by the method of Landau. It is shown that the effect of Landau damping is to make the Mach cone more diffuse, rather than to cause the wake to decay exponentially. *K. C. Westfold* (Sydney)

551:

Sears, W. R.; and Resler, E. L. Theory of thin airfoils in fluids of high electrical conductivity. J. Fluid Mech. **5** (1959), 257–273.

The authors consider the steady flow of an incompressible electrically conducting fluid past thin cylindrical bodies in the presence of a uniform magnetic field. The effects of two field orientations (parallel and perpendicular to the uniform free stream) are analysed. For the most part, the fluid is taken to be a perfect conductor although the modifications due to finite conductivity are discussed.

The analysis is based on a small-perturbation theory. Solutions of several problems are presented including the flow past an infinite sinusoidal wall, a lifting airfoil without thickness and a symmetrical airfoil with thickness.

H. Greenspan (Cambridge, Mass.)

552:

Kulikovskiĭ, A. G. Motions with homogeneous deformation in magnetic hydrodynamics. Dokl. Akad. Nauk SSSR **120** (1958), 984–986. (Russian)

In the Lagrangian description of the motion of an infinitely conducting fluid in a magnetic field, let the position at time t of any particle, initially at the position x_i^0, be

$$x_i = M_{ij}(t)x_j^0 + M_i(t).$$

It is shown that if the magnetic field lines are not straight, then the initial density, pressure and magnetic field must have the form

$$\rho^0 = \text{const},\quad p^0 = p_{ij}x_i^0x_j^0 + p_ix_i^0 + \text{const},$$
$$H_i^0 = h_{ik}x_k^0 + h_i,$$

where p_{ij}, p_i, h_{ik}, h_i are constants.

Further, if the initial density and the thermal and electric conductivities are constant throughout the volume the solutions obtained are applicable to viscous motions of a medium with finite electric conductivity.

K. C. Westfold (Sydney)

553:

Gheorghiţă, Şt. I. Sur le mouvement stationnaire des fluides incompressibles dans les milieux poreux nonhomogènes. An. Univ "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **7** (1958), no. 17, 33–37. (Romanian. Russian and French summaries)

The author presents some new approximations for different types of flow in porous non-homogeneous media. In particular, he deals with the following cases: (a) nonexistence of free-surfaces; (b) uni-dimensional motion; and (c) observations concerning motion in non-homogeneous media. Explicit solutions are presented.

K. Bhagwandin (Oslo)

554:

Vasilak, Serdžiu [Vasilache, Sergiu]. Questions on infiltration of incompressible liquids in porous media with variable permeability, exterior to an infinite cylinder. Rev. Math. Pures Appl. **2** (1957), 475–487. (Russian)

In the present paper the author studies the general problem related to infiltration of incompressible fluids in porous media interior to an infinite circular cylindrical configuration. The permeability-coefficient is supposed to vary according to an exponential law, viz., $\mu = Ae^{-\beta r}$, where A and β are positive constants and r is the radius of the cylinder. Axially symmetric flow is considered. From the Darcy law $\mathbf{v} = \mu \operatorname{grad} p$ and the equation of motion $\operatorname{div} \mathbf{v} = 0$ the author derives the fundamental equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{\partial^2 p}{\partial z^2} + (r^{-1} - \beta)\frac{\partial p}{\partial r} = 0,$$

subject to the boundary-condition $p(z, r)|_{r=a} = f(z)$, where a = radius of tube, and $f(z)$ is a continuous function.

The author presents the solution to this problem as a corollary to the solution of a closely related, more general, elliptic partial differential equation. The methods employed to solve the Dirichlet problem are that of Amerio [cf. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4** (1943), 287] and generalizations by Krzyanski [cf. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **4** (1948), 408–416; Studia Math. **11** (1949), 95–125; MR **10**, 254; **12**, 105]. A number of inequalities are presented. Finally, the author succeeds in constructing the Green-function (kernel) for this problem by virtue of some methods expounded earlier by M. Gevrey [cf., J. Math. Pures Appl. (9) **9** (1930), 1–80]. It should be noted, however, that the solution presented is by no means amenable to numerical computation. As a matter of fact, a good deal of further work has to be carried out before explicit solutions of these complicated integral equations can be obtained. *K. Bhagwandin* (Oslo)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 159, 276, 499, 543, 547, 548, 550, 609, 615.

555:

★Jenkins, Francis A.; and White, Harvey E. Fundamentals of optics. 3rd ed. McGraw-Hill Book Company, Inc., New York–Toronto–London, 1957. vii+637 pp. $8.50.

This third edition of a well-known undergraduate textbook differs from the preceding edition mainly in the following respects: mathematical derivations are somewhat simplified in a few places by the introduction of complex notation; graphical methods for ray tracing through prisms are described and the principles of "concentric optics" are explained; greater emphasis is placed on the the notion of wave packets and line width.

This edition also includes a new set of problems, which will undoubtedly be welcomed by teachers of the subject. As in the previous editions, the experimental side of the subject is emphasized. The presentation is lucid and the reader is not burdened by excessive detail. This book remains one of the best undergraduate texts on optics.

E. Wolf (Rochester, N.Y.)

556a:

Namioka, T. Theory of the concave grating. I. J. Opt. Soc. Amer. **49** (1959), 446–460.

556b:

Namioka, T. Theory of the concave grating. II. Application of the theory to the off-plane Eagle mounting in a vacuum spectrograph. J. Opt. Soc. Amer. **49** (1959), 460–465.

The author gives a theory of concave gratings from the standpoint of geometrical and physical optics, calculating exactly the light path, including higher order image errors. The theory is compared with one developed earlier by H. Beutler, and the errors in the earlier paper are pointed out; these originated from the insufficiency of the approximations used.

The above theory is applied to an off-plane Eagle vacuum spectrograph, and calculations and measurements are made employing a grating of 21 foot radius and 30,000 lines/inch. The measurements agree very well with the calculations. *M. Herzberger* (Rochester, N.Y.)

557:

Marx, Helmut. Linearisierung der Durchrechnungsformeln für windschiefe Strahlen. II. Opt. Acta **5** (1958), 65–70. (English and French summaries)

[For part I, see Opt. Acta **1** (1954), 127–140; MR **16**, 977.] The author describes an iteration procedure applied to trace a skew ray through an optical system and compares it with a procedure suggested by W. Weinstein and those proposed by T. Smith.

M. Herzberger (Rochester, N.Y.)

558:

Maréchal, A. La diffusion résiduelle des surfaces polies et des réseaux. Opt. Acta **5** (1958), 70–74. (German and English summaries)

The quality of optical surfaces depends on the deformations from the desired form in depth which have an influence on image formation even if these deviations are small when compared to the wavelength of light. These effects are due to diffraction at the uneven surface, and one can obtain some information about this by investigating the amount of diffuse radiation emerging from the surface.

The author shows that the total diffused energy depends on the mean square of the deformations, and that the angular energy distribution gives a measure of the angular distribution of the deformations. The theory is applicable to the investigation of inaccuracies in a diffraction grating.

M. Herzberger (Rochester, N.Y.)

559:

Trevena, D. H. On space charge waves. J. Electronics Control (1) **6** (1959), 50–64.

Expressions are derived from the plasma frequency reduction factors (p) for space charge waves in magnetically focused beams in drift tubes with arbitrary, but uniform cathode flux. The results are found to depend on the magnetic field only through the cathode flux. The division of the waves into four complete sets, necessary to match the four initial conditions is shown. The expressions for p with a propagating tube are also derived including the complex values, when beam and wave velocities are close. Pierce's C factor is found to occur in the expression, but his Q factor is found to be negative and constant. (Author's conclusions.) *J. E. Rosenthal* (Passaic, N.J.)

560:

Dawson, John M. Nonlinear electron oscillations in a cold plasma. Phys. Rev. (2) **113** (1959), 383–387.

The author studies a plasma of charged particles without collisions or thermal motions, and only electrostatic interaction. For the one-dimensional case the nonlinear equations are solved exactly to give periodic solutions with the same plasma frequency ω_P that occurs in the usual linear approximation. Beyond a certain maximum value for the amplitude the solution is no longer rigorous. In the three-dimensional case cylindrical and spherical waves are found, which however have a finite lifetime. Finally an approximate method for finding more general solutions is discussed. *N. G. van Kampen* (Utrecht)

561:

Fang, P. H. Conductivity of plasmas to microwaves. Phys. Rev. (2) **113** (1959), 13–14.

Explicit calculations of the complex conductivities of a plasma with a Maxwellian distribution of electrons are made for the case where the collision cross-section is velocity independent and the case where it is inversely proportional to the velocity.

C. H. Papas (Pasadena, Calif.)

562:

Skugarevskaya, O. A. Computation of the final stage of the process of establishment of an electric field in a three-layered medium. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 59–72. (Russian)

The problem considered is the final stage of the establishment of an electric field produced by an electric dipole of d.c. current in the following system: Two homogeneous isotropic conducting layers are lying on an insulator of infinite capacity. The upper half-space (air) is also considered an insulator. The treatment consists in the solution of Maxwell's equations by previously developed mathematical techniques. [A. N. Tikhonov, Izv. Akad. Nauk SSSR. Ser. Geograf. Geofiz. **10** (1946), 213–231; **14** (1950), 199–222; MR **8**, 186; **12**, 65; A. N. Tikhonov and O. A. Skugarevskaya, ibid. **14** (1950), 281–293; MR **12**, 376.] Extensive numerical results are given in the form of curves. *J. E. Rosenthal* (Passaic, N.J.)

563:

Waldron, R. A. A helical coordinate system and its applications in electromagnetic theory. Quart. J. Mech. Appl. Math. **11** (1958), 438–461.

In the first part of this paper a non-orthogonal helical coordinate system is described in which the coordinate surfaces are right circular cylinders, r = constant, radial planes, φ = constant, and helical surfaces, $\zeta = z - p\varphi/2\pi =$ constant, the latter being generated by radial lines rotating with constant angular velocity ω and simultaneously translating with constant speed, $dz/dt = p\omega/2\pi$, in the direction of the helical axis. Some of the geometrical properties of the system are given, and the transformation of the familiar scalar and vector differential operators from cylindrical to helical coordinates is worked out. Finally, the characteristic functions and characteristic values of the time-independent wave equation are found from the usual product solution. The unusual feature of this solution is that the radial eigenfunctions involve Bessel functions of order $q = n - p\beta/2\pi$, where n and β are

the propagation constants for progressive waves in the φ and ζ directions respectively.

The second part of the paper presents fundamental helical field components for electromagnetic waves corresponding to the E and H waves of a cylindrical coordinate system. These solutions are applied to a number of physical problems including the helical wire, the slow-wave tube composed of a conducting cylinder with enclosed helical wire, the helical waveguide bounded by a pair of cylinders and a pair of helical surfaces, and the coaxial transmission line with helical dielectric support.

This paper is a model of clear, concise writing.

R. D. Kodis (Providence, R.I.)

564:

Hauser, Walter. On the theory of anisotropic obstacles in cavities. Quart. J. Mech. Appl. Math. **11** (1958), 112–118.

A description is given of the electromagnetic field within a charge and current free cavity having conducting walls, when an obstacle of anisotropic nature is enclosed. The electric and magnetic fields at any point in the cavity are related to the fields within the obstacle by employing tensor Green's functions and expansions of the latter in terms of vector eigenfunctions for the empty cavity. A system of integral equations is then secured for the fields within the obstacle, and similar relations are obtained for the adjoint fields which obey a modified form of the Maxwell equations. By manipulation of these integral equations, the difference of the square of the resonant frequencies of the empty and loaded cavity is exhibited in variational forms. Another coupling leads to a perturbation formula for the frequency shift of the Bethe-Schwinger type. Since the variational principles are not employed in any specific fashion (and are complicated by the multiple appearance of the resonant frequency), their alleged practical superiority over perturbation methods remains undocumented.

H. Levine (Stanford, Calif.)

565:

Tihonov, A. N.; and Svešnikov, A. G. Slow motion of a conducting medium in a stationary magnetic field. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1959, 49–58. (Russian)

The effects are considered which arise when a conducting medium moves in a stationary magnetic field. Relations are obtained between the velocity of flow of ocean currents and the magnitude of the electric field gradient induced by the magnetic field of the earth. These relations take into account the finite width of the flow and the finite conductivity of the ocean bottom

J. E. Rosenthal (Passaic, N.J.)

566:

Schlüter, A. Kraftfreie Magnetfelder. II. Z. Naturf. **12a** (1957), 855–859.

[For part I, see R. Lüst and A. Schlüter, Z. Astrophys. **34** (1954), 263–282; MR **17**, 110.] A magnetic field is termed force free when it does not exercise any force upon matter. This implies that in this case electric currents must be everywhere parallel to the magnetic vector. The general solution of the differential equation of force free magnetic fields is given for the case of cylindrical symmetry.

F. Oberhettinger (Madison, Wis.)

567:

Karavainikov, V. N. Amplitude and phase fluctuations in a spherical wave. Akust. Ž. **3** (1957), 165–176. (Russian)

A study is made of the inter-correlation and the longitudinal and transverse self-correlation of the amplitude and phase fluctuations in a spherical wave. The wave-optical approach is used. The calculations are based on the small perturbation method of S. M. Rytov [Izv. Akad. Nauk SSSR, Ser. Fiz. **2** (1937), 223–259; a French version appeared in Actualités Sci. et Indus. no. 613, Hermann, Paris, 1938], which does not impose any restrictions on the magnitude of the fluctuations.

J. E. Rosenthal (Passaic, N.J.)

568:

Grinberg, G. A. A method for solving problems of the diffraction of electromagnetic waves by ideally conducting plane screens, based on the study of the currents induced on the shaded side of the screen. I. General foundations of the method. Soviet Physics. Tech. Phys. **28** (**3**) (1958), 509–520 (542–554 Ž. Tehn. Fiz.).

The problem concerning the diffraction of electromagnetic waves around perfectly conducting plane screens (in $z=0$) can be reduced in various ways to the solution of integral equations for the distribution of the current density $\mathbf{j}(x, y)$ induced in these screens. This article starts from the following equation of this type [holding for time-harmonic fields proportional to $\exp(i\omega t)$]:

$$(1)\quad c^{-1}\int \mathbf{j}(Q)\left\{\frac{\exp(-ikQP)}{QP}\right\}df(Q) = -\mathbf{A}^0(P)-(ik)^{-1}\operatorname{grad} u(P);$$

Q and P represent an integration point and a point of observation in the diffracting screen (surface element df), $\mathbf{A}^0$ the vector potential of the primary field, and $u(P)$ a solution of the two-dimensional equation:

$$(2)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+k^2u = ik\left(\frac{\partial A_z^0}{\partial z}\right)_{z=0};$$

u has to be bounded and single-valued on the screen. The equations (1) and (2), combined with the condition that $\mathbf{j}$ should be directed tangentially along the screen boundaries, fix both the unknown distribution $\mathbf{j}(x, y)$ and the particular solution u of (2).

The integral equation (1) is reduced next to an equivalent one referring to the current distribution $\mathbf{j}_2$ along the shaded side of the screen (assuming, for the moment, all sources in the half space beyond the other side). The new equation reads:

$$(3)\quad c^{-1}\int \mathbf{j}_2(Q)\left\{\frac{\exp(-ikQP)}{QP}\right\}df(Q) = (4\pi)^{-1}\int [\mathbf{i}_z\cdot H^0]_Q\left\{\frac{\exp(-ikQP)}{QP}\right\}df'(Q)-(ik)^{-1}\operatorname{grad} W',$$

in which df' represents the surface elements of $z=0$ that are not occupied by the screen, $\mathbf{i}_z$ a unit vector in the z direction, H^0 the primary magnetic field, and W' a solution of the homogeneous equation corresponding to (2). The boundary condition fixing both j_2 and W', in connection with (3), reads $j_{2,n} = -(c/4\pi)H_s^0$, the left hand side being the current component perpendicular to the screen boundary, H_s^0 the tangential component there of the

primary magnetic field. The equation (3) proves to be more convenient than (2).

The author is also concerned with a set of diffracting screens in parallel planes; it leads to simultaneous integral equations of the type (3). The case of two-dimensional fields characterized by $\partial/\partial y = 0$ is worked out in particular. The role of the kernel $\exp(ikQP)/QP$ is then taken over by the Hankel function $H_0^{(2)}(QP)$. The two-dimensional problem dealing with n parallel diffracting strips at $z = z_s$, $a_s < x < b_s$, $-\infty < y < \infty$ $(s = 1, 2, \cdots, n)$ can be reduced, with the aid of quadratures, to the solution of the set of integral equations

$$(4)\quad \sum_{s=1}^{n} \int_{a_s}^{b_s} I^{(s)}(\xi, R) \cdot H_0^{(2)}(k|x_p - \xi|)d\xi = -H_0^{(2)}(k|x_p - R|),$$

in which $a_p < x_p < b_p$ and $p = 1, 2, \cdots, n$. The solution of this "key problem" is equivalent to the determination of the current distributions on the strips in question the field is generated by a homogeneous line source along $x = R$, $z = 0$. *H. Bremmer* (Eindhoven)

569:

Grinberg, G. A. A method for solving problems of the diffraction of electromagnetic waves by ideally conducting plane screens, based on the study of the currents induced on the shaded side of the screen. II. Diffraction by a half-plane and by a plane with an infinitely long straight slit. Soviet Physics. Tech. Phys. **28** (3) (1958), 521–534 (555–568 Ž. Tehn. Fiz.).

The theory of part I of this article [reviewed above] is applied to the examples mentioned in the subtitle. The diffraction by a half plane $(x > 0, z = 0$; Sommerfeld problem) is first discussed for sources admitting the two-dimensional version characterized by $\partial/\partial y = 0$. This special case can further be reduced to the two subcases in which either the electric or the magnetic field has a non-vanishing component in the y direction only. The associated "key problem" (see the review of part I) concerns the determination of the current distribution $j_y(x)$ produced at the shadow side $(x > 0; z = -0)$ by an infinite line source along $x = -R$, $z = 0$. The possibility of solving the basic equation of this key problem (equation (4) in the review of part I) is connected with the identity:

$$(1)\quad (R^{1/2}/\pi) \int_0^\infty \exp\{-ik(\xi + R)\} \cdot (R + \xi)^{-1}\xi^{-1/2} H_0^{(2)} \times (k|x - \xi|)d\xi = H_0^{(2)}\{k(x + R)\} \quad (x > 0,\ R > 0)$$

and with a limiting form of the latter for $R \to \infty$. The complete current distribution in the screen can then be expressed analytically in terms of the distribution of the primary magnetic field over the complementary half plane $x < 0$, $z = 0$.

The Sommerfeld problem is discussed next for arbitrary sources not involving $\partial/\partial y = 0$. The Fourier transform with respect to y of the integral equations for the current distribution on the shadow side leads to another "key problem", which can be solved, once again, with the aid of (1). Consequently, it also proves to be possible to obtain this current distribution for an arbitrary distribution of the primary magnetic field over the complementary half plane $x < 0$, $z = 0$.

The explicit discussion of the diffraction through a straight slit (at $-a < x < a$, $z = 0$) is restricted to the case $\partial/\partial y = 0$. The determination of the current distributions along the shadow sides $(x < -a, z = -0$ and $x > a, z = -0)$, $j_z^{(l)}(x)$ and $j_z^{(r)}(x)$ say, is reduced to that of the functions

$$w_\pm(r) = (4\pi^2/c)\sqrt{r}\exp\{ika(r+1)\} \{j_z^{(r)}(a + ar) \pm j_z^{(l)}(-a - ar)\}.$$

In the case of an electric field in the y direction these functions $w_\pm$ are to be solved from the set of integral equations

$$(2)\quad w_\pm(r) = F_\pm(r) \mp \{\exp(-2ika)/\pi\} \times \int_0^\infty w_\pm(\rho)(\rho + 2/\rho)^{1/2}(r + \rho + 2)^{-1}\exp(-2ika\rho)d\rho \quad (r > 0),$$

the quantity $(\rho + 2/\rho)^{1/2}$ has to be replaced by $-[\rho/(\rho+2)]^{1/2}$ in the other subcase of a magnetic field in the y direction. The functions $F_\pm(r)$ depend on the given distribution of the primary magnetic field over the slit. Solving of (2) by successive approximations proves to be convenient apart from $ka \gg 1$, even for values of ka of the order of unity. The special case $\lambda = 2a$ is also discussed numerically. *H. Bremmer* (Eindhoven)

570:

Kazarinoff, N. D.; and Ritt, R. K. On the theory of scalar diffraction and its application to the prolate spheroid. Ann. Physics **6** (1959), 277–299.

The authors solve the problem of finding the asymptotic distribution on the surface of a prolate spheroid of the field produced by a plane wave whose direction is along the major axis of the spheroid. The result is asymptotic for large w, the circular frequency of the incident wave. A general method, described at first, presupposes any source ρ and a coordinate system in which the surface of the diffracting body is a coordinate surface and in which the reduced wave equation, $\nabla^2 v + (w - is)^2 v = \rho$ is separable. An integral representation for v is sought. By assuming that the source is symmetric with respect to one independent variable the problem reduces to two independent variables. Then the operator in the partial differential equation is treated as the sum of two operators each involving one independent variable. The theory of resolvents for linear operators developed by Phillips [Math. Res. Group. New York Univ. Res. Rep. no. **EM-42**; MR **14**, 1088] and Sims [J. Math. Mech. **6** (1957), 247–285; MR **19**, 144] is applied to find the one-dimensional resolvent Green's functions.

The author applies the method to the problem of the prolate spheroid. The integral representation for the field v is obtained and then specialized to a plane wave coming in along the major axis. The integral representing v is now to be evaluated only on the surface of the spheroid. This integral is along a path in the complex plane and so is to be evaluated by residues. The singularities of the integral are determined by those solutions of two ordinary differential equations which enter into the resolvent Green's functions. To obtain these singularities the condition that w be large is applied and Langer's method is used to obtain asymptotic solution for the two ordinary differential equations. The residues of the integral are stated and the resulting value of the integral interpreted as creeping waves. *M. Kline* (Aachen)

571:

Keller, Joseph B.; Lewis, Robert M.; and Seckler, Bernard D. Diffraction by an aperture. I, II. J. Appl. Phys. **28** (1957), 426–444, 570–579.

Identical with two reports [Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-92**, **EM-96** (1956); MR **20** #5033, #5034], the first by Keller alone.

572:

Saiasov, Iu. S.; and Sinitsyna, Iu. V. On the theory of concave waveguides. Soviet Physics. Tech. Phys. **28** (3) (1958), 1202–1208 (1293–1300 Ž. Tech. Fiz.).

The author studies the propagation of transverse electric waves in a waveguide whose cross section is the simply connected symmetrical region bounded by an ellipse and a hyperbola with the same foci. The field is described by the membrane function for the region which satisfies Neumann boundary conditions. The Mathieu equations, resulting from separation in elliptic coordinates, are solved by series expansions under the condition that the distance between the vertices of the hyperbola is small compared to the semi-axes of the ellipse.

R. N. Goss (San Diego, Calif.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 522, 523, 525, 527.

573:

Christie, Dan E. Some thermodynamic properties of a mapping. J. Franklin Inst. **267** (1959), 119–133.

The author examines systematically the consequences of the fact that the area enclosed by a curve representing a reversible cycle is the same in a (P, v) and a (T, s) diagram, as is well known. It is easy to guess that this enquiry cannot lead to results which had not been discovered before or which are of more than marginal physical importance. The reviewer agrees with the author that "teachers and students of the subject will find some pleasure ··· and some amusement in following through [the] arguments ···".

Essentially the same ideas, but admittedly deprived of the mathematical language of projective geometry used here, have been introduced into thermodynamics by W. Nusselt some fifty years ago. The reviewer believes that Nusselt was first to introduce the method of deriving thermodynamic relations, such as e.g. the Maxwell relations, by the consideration of elementary cycles and by the use of the above property of the areas enclosed by them. This method of derivation, usually longer and less natural than analytic methods, is said to have a special appeal to "engineers".

The style of the paper is unnecessarily ponderous.

J. Kestin (London)

574:

Vodička, Václav. Some problems on heat conduction in stratiform bodies. J. Phys. Soc. Japan **14** (1959), 216–224.

A boundary value problem for one-dimensional heat conduction in a semi-infinite solid of n layers of different material is analyzed with the aid of the Laplace transformation and matrix algebra. Each layer has a uniform initial temperature. The temperature of the outer face is a simple harmoic function of time. The difficulties of solving for the temperature function in the general case are pointed out. In some special cases explicit formulas for the temperature function $u(x, t)$ are obtained.

R. V. Churchill (Ann Arbor, Mich.)

575:

★Sparrow, E. M.; and Siegel, R. Thermal entrance region of a circular tube under transient heating conditions. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 817–826. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The paper treats the problem of the thermal entrance region of a circular pipe with a fully developed parabolic profile. An arbitrary variation of temperature along the wall with time is expressed in terms of step-changes, and the fundamental problem of the response to a step change in temperature is solved by the use of the energy integral equation and a cubic polynomial assumption for the velocity profile at a cross-section. The resulting equation for the dimensionless thermal boundary layer thickness Δ is solved by the method of characteristics. The analysis is repeated for arbitrarily prescribed heat fluxes at the wall. The results are compared with known limiting cases and the good agreement obtained lends support to the assumptions.

J. Kestin (London)

QUANTUM MECHANICS

See also 156, 289, 418, 460, 461, 619.

576:

Riazanov, G. V. Quantum-mechanical probabilities as sums over paths. Soviet Physics. JETP **35** (8) (1959), 85–93 (121–131 Ž. Eksper. Teoret. Fiz.).

A new method for formulating non-relativistic quantum mechanics is proposed. In appearance it resembles the Feynman principle [R. P. Feynman, Rev. Mod. Phys. **20** (1948), 367–387; MR **10**, 224] but differs from it in two important respects: (i) it gives a direct expression for the probability rather than for the probability amplitude; (ii) in its sum over histories the possibility of a change in the sign of time along the path is incorporated. The fundamental postulate is that the probability for finding a given value a for a physical quantity $a\{x(t)\}$, which depends on the path $x(t)$, is given symbolically by

$$W(a) = \int_a \cos\frac{S}{\hbar}\,d\Gamma. \tag{1}$$

$\int d\Gamma$ denotes the integral over all paths, in the sense of (ii) above, for which $a\{x(t)\}=a$, S is the classical action of the path $x(t)$, and $\hbar$ is Planck's constant.

It is clear from the form of (1) that there is some danger of probabilities being negative, but some cases of physical interest are analysed and it is shown that in these cases the probabilities can be written as products of amplitudes which are manifestly positive.

The description of the temporal development of events poses problems in this formalism and these are discussed in terms of the behaviour to be postulated in the infinite past and the infinite future.

While the principle reproduces many of the accepted features of conventional quantum mechanics it is claimed that it is more general in that (1) does not always permit decomposition into probability amplitudes and so can apply to situations in which quantum mechanics would be meaningless.

Throughout an analogy is stressed with Gibbs' principle in statistical mechanics.

J. C. Polkinghorne (Cambridge, England)

577:

Bodiou, G. Probabilité sur un treillis non modulaire. Publ. Inst. Statist. Univ. Paris **6** (1957), 11–25.

The author gives an axiomatic treatment of probability in quantum mechanics. Starting with the usual assumption that statements A, B, $\cdots$ form an orthocomplemented, semi-modular lattice L (for example, the lattice of all closed subspaces of Hilbert space), he defines a probability as a non-negative valuation which is real, additive on orthogonal elements, and satisfies $p(I)=1$. He then obtains conditions for the probability to be of "classical" type, and for L to be "atomic". He also axiomatizes conditional probabilities, and considers some implications of modularity.

G. Birkhoff (Cambridge, Mass.)

578:

Jauch, J. M. Theory of the scattering operator. II. Multichannel scattering. Helv. Phys. Acta **31** (1958), 661–684.

This is a generalization of the author's theory of simple scattering systems [same Acta **31** (1958), 127–158; MR **20** #682] to multi-channel systems. The point is again to put the well-known quantum mechanical scattering theory on a rigorous mathematical basis. The various channels α are characterized by unitary operators

$$U_t^{(\alpha)} \equiv \exp(-iH_\alpha t) \quad (-\infty < t < \infty)$$

which commute with each other for all α and t. The total system is defined by $V_t \equiv \exp(-iHt)$ where H is the total energy operator. It is assumed that the

$$\lim_{t\to\pm\infty} V_t^* U_t^{(\alpha)} f = f_\pm^{(\alpha)}$$

exists in the strong topology. The corresponding ranges $R_\pm^{(\alpha)}$ are shown to be (infinite dimensional) subspaces. Let $R_\pm$ be closure of the set of all these subspaces. The basic assumption is then that $R_+ = R_- = N$, where N is the subspace of all continuum states of H. Channel wave operators $\Omega_\pm^{(\alpha)}$ are defined and the scattering operator $S=\sum_\alpha \Omega_+^{(\alpha)}\Omega_-^{*(\alpha)}$ is proven to exist, to be unitary in N, and to annihilate $N^\perp$. The operator $\Omega_-^*\Omega_+$ defined in the previous paper is not generalizable to multi-channel systems. The "in" and "out" operators are discussed in the last section.

F. Rohrlich (Baltimore, Md.)

579:

Fierz, M. Die Anzahl der mit Hilfe von $2l$ Dirac'schen Spinoren zu bildenden Invarianten. Helv. Phys. Acta **31** (1958), 587–590.

The paper is an extension of Pauli's result [Zeeman-Verhandelingen, Haag, 1935, p. 31] which states that one can form five scalar invariants out of the four Dirac spinors. The extension says that there are also five pseudoscalar invariants which can be formed from the four Dirac spinors. The van der Waerden spinor calculus is used to prove the result. Finally the theorem is even further extended and it is shown that out of $2l$ spinors one can construct

$$\frac{(2l)!\,2^{l+1}1\cdot 3\cdot 5\cdots(2l+1)}{l!\,(l+1)!\,(l+2)!}$$

invariants.

M. J. Moravcsik (Livermore, Calif.)

580:

Zemach, Ch.; and Klein, A. The Born expansion in non-relativistic quantum theory. Nuovo Cimento (10) **10** (1958), 1078–1087. (Italian summary)

It is shown that if a non-relativistic potential satisfies certain stated boundedness conditions it possesses at sufficiently high energies a convergent Born series for its Green's function and its wave functions and scattering amplitudes. In the high energy limit these series are approximated by their leading terms.

J. C. Polkinghorne (Cambridge, England)

581:

Crawford, J. A. An alternative method of quantization: the existence of classical fields. Nuovo Cimento (10) **10** (1958), 698–713. (Italian summary)

A new form of quantization is proposed in which states are represented by self-adjoint linear operators. The scalar product is defined by

$$(M, N) = \text{Trace } M^\dagger N,$$

and the normalizable linear operators form an associated Hilbert space. The self-adjoint operators on this Hilbert space (called superoperators and to be denoted by small letters) are the observables of the theory. For a given A belonging to the Hilbert space we may define two superoperators by

$$a^l M = AM, \quad \text{all } M;$$
$$a^r M = MA, \quad \text{all } M.$$

A corresponding classical observable may be defined by $\frac{1}{2}(a^l + a^r)$. If the a^l's and a^r's are canonical quantum variables the corresponding classical variables all commute. Thus this form of quantization retains classical observables along with the quantum observables and these classical observables are related to the question of hidden deterministic parameters.

All observable predictions of quantum theory are identically reproduced and no new results are obtained.

J. C. Polkinghorne (Cambridge, England)

582:

Pócsik, G. On the integrals of motion of the generalized Dirac-equation of Rayski. Acta Phys. Acad. Sci. Hungar. **8** (1958), 277–283.

The author shows that the total angular momentum for the generalized Dirac equation of Rayski [Acta Phys. Polon. **15** (1956), 89–109; MR **18**, 260]:

$$\left(\sum_{\mu=0}^{3} \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + a\frac{\partial}{\partial r^\mu}\right) + \chi\right)\psi(x, r) = 0$$

(γ the usual 4×4 Dirac matrices; χ, a real number; x, space-time coordinate; r, "internal coordinate") is the sum of a spin part, an orbital part in x-space and an orbital part in p-space.

A. S. Wightman (Princeton, N.J.)

583:
Eriksson, H. A. S. On spinor wave equations containing reflection operators. Ark. Fys. **15** (1959), 31–32.

"An example is given of a system of two first order wave equations containing a reflection operator. Applied to an electron in a Coulomb field, the ordinary fine structure formula is obtained." *Author's summary*

584:
Wataghin, G. Causality, complementarity and S-matrix formalism in 4 non-local relativistic theories of fields. Nuovo Cimento (10) **9** (1958), 519–523.

The author considers the Lee model field theory [T. D. Lee, Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317] using at the outset the relativistic invariant cut-off operators introduced in his previous work [G. Wataghin, Nuovo Cimento (10) **5** (1957), 689–701; MR **19**, 220]. The eigenvector for the "physical" V-particle is obtained rigorously (as in Lee's work), as are the $\theta-N$ and $\theta-V$ cross sections.

A general picture of S-matrix processes in non-local field theories is then proposed. The "physical" state represents a particle not interacting with other particles. Thus the asymptotic incoming and outgoing waves in the S-matrix are "physical" particles. In the small domain of non-locality where particles interact, bare particles are created and destroyed. A bare particle created in this fashion is modified into a "physical" particle as it leaves the interaction region. For the Lee model discussed in the paper, it is shown that macroscopic causality is preserved.
R. Arnowitt (Syracuse, N.Y.)

585:
Raševskiĭ, P. K. On the mathematical foundations of quantum electrodynamics. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 3 (81), 3–110. (Russian)

The author describes the mathematical apparatus of the theory of a charged spin-one-half-field and the electromagnetic field (only the uncoupled case is considered apart from a few remarks at the end of the article). The treatment is directed to readers who like to see all details spelled out. [The author is apparently unaware of previous accounts of the theory of free fields which also satisfy all requirements of mathematical rigor. See, for example, J. M. Cook, Trans. Amer. Math. Soc. **74** (1953), 222–245; MR **14**, 825; D. Kastler, Ann. Univ. Sarav. **5** (1956), 186–203, 204–227; MR **18**, 973.] For the spin-one-half-field the author uses the van der Waerden notation of dotted and undotted indices and presumes an acquaintance with a previous article [P. K. Raševskiĭ, Uspehi Mat. Nauk **10** (1955), no. 2 (64), 3–110; MR **17**, 124] on spinors. All operators and states are realized in a covariant momentum configuration space (Fock space); for the photons the indefinite metric is treated in detail. The topics discussed include one photon states, many photon states, photon annihilation and creation operators in x and p space, field operators energy and momentum of the electromagnetic field and a parallel treatment for the spin-one half particles.
A. S. Wightman (Princeton, N.J.)

586:
Havas, Peter. Equations of motion of point particles in fields of nonzero rest mass and spin. Phys. Rev. (2) **113** (1959), 732–740.

The divergence free conditions imposed by Harish-Chandra [Proc. Roy. Soc. London Ser. A **185** (1946), 269–287; MR **7**, 538] for the sources of fields of arbitrary spin are replaced by less restrictive conditions. Equations of motion are obtained for the sources when they are multipole singularities. Both neutral and charge-symmetric fields are discussed.
J. C. Polkinghorne (Cambridge, England)

587:
Nakano, Tadao. Quantum field theory in terms of Euclidean parameters. Progr. Theoret. Phys. **21** (1959), 241–259.

The unitary trick for obtaining representations of the Lorentz group and the recently discovered analyticity properties of quantum field theory [A. S. Wightman, Phys. Rev. (2) **101** (1956), 860–866; MR **18**, 781] both lead to an intimate connection between Minkowski space and a four-dimensional Euclidean space. The author considers the possibility of formulating quantum field theory in the Euclidean space and of subsequently obtaining the physical theory by analytic continuation. A variational principle is set up but no discussion is made of how this leads to commutation relations. (The essential difference between elliptic and hyperbolic equations might lead to an interesting problem here.) The author considers that the Euclidean formulation may prove superior when non-local interactions are considered.
J. C. Polkinghorne (Cambridge, England)

588:
Zimmermann, W. On the bound state problem in quantum field theory. Nuovo Cimento (10) **10** (1958), 597–614. (Italian summary)

The author discusses the theory of a single scalar field whose states involve two stable spin zero particles of distinct non-vanishing rest masses, which are conventionally designated the elementary particle (mass m) and the composite particle (mass M). Besides the usual requirements of Lorentz covariance, absence of negative energy states and microscopic causality, he assumes that the field operators $A(x)$ form an irreducible operator ring. He constructs the S-matrix in terms of τ-functions (vacuum expectation values of T-products of $A(x)$) and shows that the expression may be made symmetric in the two types of particle by associating with the composite particle a field operator $B(x)$ which is defined as the limit of a polynomial in T-products of $A(x)$. The operator $B(x)$ is local and invariant, just as $A(x)$ is, so the principle of microscopic causality provides no way of distinguishing elementary from composite particles. However, the theory discussed here may differ essentially from the case of two particles associated with two distinct fields on account of the dependence of $B(x)$ upon $A(x)$.

The bound state problem has been described in similar terms by Nishijima [Phys. Rev. (2) **111** (1958), 995–1011; MR **20** #3007]. The relation of this approach (weak operator convergence) to the method of strong operator convergence has been discussed by Haag [Phys. Rev. **112** (1958), 669–673; MR **20** #6296]. *P. W. Higgs* (London)

589:
Yappa, Yu. A. On the method of functionals in the quantum field theory. Scalar field. Vestnik Leningrad. Univ. **13** (1958), no 22, 172–181. (Russian. English summary)

In Fok's theory of quantised fields, which are regarded

as consisting of variable numbers of particles described by a set of functions $\{\psi_0, \psi_1(x), \psi_2(x_1, x_2), \cdots\}$ or by a functional

$$\Omega(\varphi) = \sum \frac{1}{\sqrt{n!}} \int \psi_n(x_1, x_2, \cdots, x_n) \bar{\varphi}(x_1) \cdots \bar{\varphi}(x_n) dx_1 \cdots dx_n$$

defined for ϕ in a Hilbert space, the problem arises as to what functionals can be so represented. The author points out that the theorem that every bounded weakly differentiable functional in an open set is analytic in that set leads to the conclusion that every such functional Ω has this representation. The Ω can be normed by the formula

$$\|\Omega\|^2 = \sum \int \psi_n \bar{\psi}_n dx_1 \cdots dx_n.$$

Implications for the theory of second quantisation are discussed. *J. L. B. Cooper* (Cardiff)

590:

Freistadt, Hans. Poisson brackets in field theory. Canad. J. Phys. **37** (1959), 5–9.

591:

Pekeris, C. L. Ground state of two-electron atoms. Phys. Rev. (2) **112** (1958), 1649–1658.

A method is given for the expansion of the solution of the wave equation for a two-electron atom in a set of orthogonal functions in perimetric co-ordinates. The series of functions chosen contains a fixed exponential term of the variable parameters of the Hylleraas type solution. The advantages claimed are that in the set of equations for determining the coefficients in the expansion, all the matrix elements are integers, and the matrix is of band type; this permits the use of a large number of rows without exhausting the storage capacity of the computing machine used for the calculation. Details are given of the method used for determining the eigenvalues of the resulting matrix. Accurate values are obtained for the ionization potential, including corrections for the mass polarization and relativistic effects and for the Lamb shift, for atomic number $Z=2$ to 10. *D. F. Mayers* (Oxford)

592:

Mittleman, Marvin H.; and Watson, Kenneth M. Scattering of charged particles by neutral atoms. Phys. Rev. (2) **113** (1959), 198–211.

Techniques based on the algebraic manipulations of the formal theory of scattering and constructs similar to those that have been applied to high energy nuclear interactions [K. M. Watson, same Rev. **105** (1957), 1388–1398; MR **18**, 853] are here applied to the scattering of a charged particle by the Coulomb field of nucleus and electrons. The state-vector of the system is written as $\Psi = F\Psi_i$, where Ψ_i describes elastic (coherent) scattering and the integral operator $(F-1)$ has matrix elements for inelastic scattering only. Ψ_i is determined by the "optical potential", $\mathscr{V}$, which in turn is related in a formally exact expression to F. The further emphasis is on the construction of $\mathscr{V}$ for which adiabatic and high-energy approximations are discussed and criteria of applicability derived. The methods are applied in detail to the scattering of an electron by hydrogen. Finally, several variational principles for the construction of $\mathscr{V}$ are discussed.

A. Klein (Philadelphia, Pa.)

593:

Osborn, R. K.; and Klema, E. D. Dynamics of an elastic ellipsoid. Nuovo Cimento (10) **9** (1958), 791–812.

An earlier series of papers by the same authors [Phys. Rev. **100** (1955) 822–834; **103** (1956), 833–834; Nucl. Phys. **2** (1956–57), 454–475; **3** (1957), 571–584] was addressed to the problem of "correlating a relatively wide range of nuclear data", in terms of a model of the nucleus in which a single particle coupled to a rigidly rotating core. The present paper is devoted to the explicit formulation of the dynamics of this coupled system, in particular the development of the kinetic energy operator. The relation of this model to other models as well as the degree to which this dynamical model is adequate to describe the nuclear model implicit in their previous work, is discussed.

E. C. G. Sudarshan (Cambridge, Mass.)

594:

Kulakov, Iu. I. Application of matrix polynomials to determine scattering phases. Soviet Physics JETP **6** (1958), 391–400.

A typical scattering matrix element for one-particle scattering or its equivalent can be expanded in generalized matrix polynomials. The expansion coefficients are simply related to the phase shifts. Parity conservation and no spin change during the collision are assumed. Consider the eigenfunctions of $\mathbf{J}^2$, $\mathbf{J}_z$, $\mathbf{L}^2$, $\mathbf{S}^2$ (associated quantum numbers JMLS) depending on a spatial direction $\mathbf{n}$ and a spin variable λ. A generalized matrix polynomial is defined as

$$\sum_M \langle \lambda'\mathbf{n}'; JMS'L' | LSMJ; \mathbf{n}\lambda \rangle.$$

Each term is simply expressible in terms of Clebsch-Gordan coefficients and spherical harmonics. Nucleon-antinucleon scattering in second order pseudo-scalar theory is used as an example. *F. Rohrlich* (Baltimore, Md.)

595:

Massey, H. S. W.; and Ridley, R. O. Application of variational methods to the theory of the scattering of slow electrons by hydrogen molecules. Proc. Phys. Sect. A. **69** (1956), 659–667.

The variational method of Hulthen [Kungl. Fysiogr. Sällsk i Lund. Förh. **14** (1944), no. 21; MR **6**, 111] and Kohn [Phys. Rev. (2) **74** (1948), 1763–1772] are applied to the calculation of the elastic scattering of slow electrons by hydrogen molecules. Polarization is neglected as is indicated by the trial wave function

$$\Psi = 3^{-1/2} \sum \psi(r_j, r_k) F(r_i) \alpha_i \chi_{jk}.$$

Here $\psi\chi$ is the ground state wave function from the hydrogen molecule as given by Coulson who employed the self-consistent field method. F, the spatial wave function for the incident electron, is taken to be a function of the spheroidal coordinate ξ equal to the sum of the distances of the nuclei from the electron divided by the distance between the nuclei. A simple form is chosen for F so that F reduces to the appropriate form for large values of ξ departing exponentially from this form as ξ approaches unity. Fairly good agreement with experiment was obtained when exchange was included.

H. Feshbach (Cambridge, Mass.)

596:

★**Longuet-Higgins, H. C. Recent developments in molecular orbital theory.** Advances in chemical physics, Vol. I, edited by I. Prigogine, pp. 239–265. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1958. xi+414 pp. $11.50.

After a brief introductory section on the method of molecular orbitals, including a discussion of Roothaan's self-consistent field procedure, the author analyses in some detail the semi-empirical theory of the spectra of aromatic hydrocarbons. The essential features of this theory are the assumption of zero differential overlap and the empirical evaluation of certain energy integrals (usually from atomic spectral data). If these two features are incorporated in the theory a very satisfactory qualitative and quantitative account of hydrocarbon spectra is obtained either by a self-consistent field calculation or by a calculation which considers limited configuration interaction (singly excited configurations only). Various attempts to justify these two basic assumptions from a theoretical point of view are discussed. In the concluding sections the spectra of more complex molecules and the free-electron molecular orbital theory are discussed briefly.

A. C. Hurley (Melbourne)

597:

Gioumousis, George; and Curtiss, C. F. Molecular collisions. I. Formal theory and the Pauli principle. J. Chem. Phys. **29** (1958), 996–1001.

The formal theory of Gell-Mann and Goldberger is used to obtain expressions for the differential cross sections describing collisions between two molecules with internal structure. These expressions are in terms of the asymptotic forms of the solutions of certain integral equations. The exclusion principle is satisfied to an approximation consistent with the Born-Oppenheimer approximation.

A. C. Hurley (Melbourne)

598:

Bourret, Richard. Particle equations from non-associative algebras. Canad. J. Phys. **37** (1959), 183–188.

The suggestion is made that, as a generalization of the quaternion algebra, an eight unit non-associative algebra of Cayley and Dickson could be used to generate relativistic wave equations. As a sample, a bilocal generalization of the Lee-Yang neutrino equation is exhibited, and some of its properties discussed. It is not mentioned whether this equation is irreducible or not.

C. A. Hurst (Adelaide)

599:

Shelepin, L. A. On the theory of high-spin particles. Soviet Physics. JETP **34 (7)** (1958), 1085–1092 (1574–1586 Ž. Eksper. Teoret. Fiz.).

The problem is considered of finding all equations of the form $(\alpha_k\partial/\partial x_k+m)\psi=0$, where ψ transforms according to a finite dimensional irreducible representation of the Lorentz group and is associated with a non-degenerate real Lagrangian and a positive definite energy density (charge) for particles with integral (half-integral) spin. The method is based on the direct products of generalized Dirac algebras

$$U(\Gamma),\quad \Gamma_i\Gamma_k+\Gamma_k\Gamma_i=2p\delta_{ik}$$

with $p\alpha$ complex number. Explicit expressions in terms of the Γ_i are obtained for the infinitesimal rotation matrices, the metric matrix, the reflection matrix, and the transformation matrix for ψ. The cases of spin 1 and spin 3/2 are treated as examples. The advantages of the present method over other methods are pointed out.

F. Rohrlich (Baltimore, Md.)

600:

Fronsdal, C. On the theory of higher spin fields. Nuovo Cimento (10) **9** (1958), supplemento, 416–443.

A simplified version of the Fierz-Pauli [M. Fierz and W. Pauli, Proc. Roy. Soc. London Ser. A **173** (1939), 211–232; MR **1**, 190] theory of relativistic wave-equations for particles of higher spin is developed, starting from the Rarita-Schwinger formulation [W. Rarita and J. Schwinger, Phys. Rev. (2) **60** (1941), 61]. A spin projection operator is defined and displayed which selects from the tensor (integral spin) or spin-tensor (half-integral spin) employed in the Rarita-Schwinger formulation some one of the definite spins contained therein. This projection obeys a wave equation which is non-linear in the four-momentum. A linearized form of the theory (for example for the half-integral spin case) is next constructed and the algebra of the ingredient matrices obtained from the requirements of definite spin and inversion symmetry. In this version the introduction of electromagnetic interaction can be carried through. (The consistent quantization of any theory with interaction and spin greater than unity has yet to be demonstrated, however.) The utility of the formulation is finally illustrated by computation of the magnetic moment for an arbitrary half-integral spin particle, shown to have a gyromagnetic ratio equal to the reciprocal of its spin, and by the computation of angular correlation distributions for the decay of spin 3/2 particles.

A. Klein (Philadelphia, Pa.)

601:

Ouchi, Tadashi. Mass reversal and weak interactions. Progr. Theoret. Phys. **20** (1958), 909–919.

This is the most comprehensive paper belonging to the sequence of mass-reversal theories all of which, except for the original paper of J. Tiomno [Nuovo Cimento (10) **1** (1955), 226–232; MR **16**, 1184], "derive" the Universal $V-A$ four-fermion interaction. In addition to the mass reversal transformation for each individual field, which is here called $MR(1)$, the author considers the combined mass reversal of 2, 3 and 4 fields. The consequences of these more general mass reversal transformations are examined and it is shown that with a suitable choice of a "mass parity type" several unwanted processes can be forbidden. Interaction terms with coupling constants dependent on the particle masses are also allowed in this scheme.

E. C. G. Sudarshan (Cambridge, Mass.)

602:

Aviles, Joseph B., Jr. Extension of the Hartree method to strongly interacting systems. Ann. Physics **5** (1958), 251–281.

A trial function of the form

$$\psi(r_1,\cdots,r_N)=\Phi(r_1,\cdots,r_N)\prod_{i<j=1}^{N}f(r_i-r_j)$$

is used to describe the ground state of a many-body system with strong interactions. Variational conditions are derived for the function $f(r_i-r_j)$ which is zero for

$|r_i - r_j| < r_0$ and approaches unity for $|r_i - r_j| \gg r_0$: the product over all pairs is a measure of the correlation due to strong forces outside the hard cores. Both Bose systems and Fermi systems are considered. In the case of the former Φ is simply unity, whereas in the latter it is a Slater determinant of single-particle functions with the required antisymmetry properties.

For Bose systems the expectation value of the energy is reduced to simple form by the application of the cluster expansion method developed in classical statistical mechanics for the imperfect gas. The variational condition determines $f(r_{ij})$; it is found to satisfy an integro-differential equation resembling the Schrödinger equation for two particles with a potential term consisting of the actual two-body interaction plus an effective interaction with neighbouring particles. This equation represents an extension of the Hartree equation to strongly interacting systems. A simpler, less general method is also derived and applied to obtain an analytic solution for the low density, hard sphere Boson case. Results are in good agreement with non-variational calculations.

The method is extended to Fermi systems. Here the expectation value of the energy depends also on the Slater determinant and more general methods are required to obtain a cluster expansion. The results are applied to the hard sphere, Fermi gas at low density.

C. Froese (Vancouver, B.C.)

603:

Pavlikovskii, A.; and Shchuruvna, V. On Zubarev's method of auxiliary variables in statistical physics. Soviet Physics. Dokl. **118** (**3**) (1958), 71–74 (61–64 Dokl. Akad. Nauk SSSR).

Zubarev's method of auxiliary variables which is equivalent to the Bohm and Pines method is used to evaluate the partition function for a system of interacting fermions.

D. ter Haar (Oxford)

604:

Bogoliubov, N. N. On a variational principle in the many-body problem. Soviet Physics. Dokl. **119** (**3**) (1958), 292–294 (244–246 Dokl. Akad. Nauk SSSR).

A system of interacting fermions is considered in second quantization formalism. After a canonical transformation of the second quantization operators the resultant energy is minimized with respect to the parameters entering into the transformation equations. It is shown that the Fock solution is one of the solutions of the variational equations.

D. ter Haar (Oxford)

605:

Kvasnikov, I. A.; and Tolmachev, V. V. On a variational principle in the statistical problem of many bodies. Soviet Physics. Dokl. **120** (**3**) (1958), 553–557 (273–276 Dokl. Akad. Nauk SSSR).

The variational method introduced by Bogolyubov in the paper reviewed above is extended to a consideration of excited states and of the partition function of a system of interacting fermions. *D. ter Haar* (Oxford)

606:

Hiroike, Kazuo. On the method of "auxiliary variables". Progr. Theoret. Phys. **21** (1959), 327–342.

The author is concerned with the method of definition of collective coordinates (auxiliary variables) and the Hamiltonian operator in a system of a finite number of bosons. His theory is a revision of that given by N. N. Bogolyubov and D. N. Zubarev [Ž. Eksper. Teoret. Fiz. **28** (1955), 129–139; MR **17**, 113].

E. L. Hill (Minneapolis, Minn.)

607:

Trifonov, E. D. Construction of many-electron coordinate functions. Vestnik Leningrad. Univ. **13** (1958), no. 22, 42–47. (Russian. English summary)

Young's diagrams and the projection operators of the regular representation of the symmetric group are used for the study of the wave function of a system of many electrons with fixed total spin. The discussion is similar in principle to that given by F. Hund [Z. Physik **43** (1927), 788–804]. *E. L. Hill* (Minneapolis, Minn.)

608:

Haug, Albert; und Sauermann, Günter. Untersuchungen zur adiabatischen Näherung bei Festkörperproblemen. Z. Physik **153** (1958), 269–277.

Von M. Born und R. Oppenheimer [Ann. Physik **84** (1927), 457–484] wurde zuerst die adiabatische Näherung in die Theorie der Materie (in der zitierten Arbeit für Moleküle) eingeführt. Nach dieser Auffassung sind für die Elektronenbewegung nur die momentanen Lagen der (um ihre Gleichgewichtslagen schwingenden) Gitterteilchen, jedoch nicht deren Geschwindigkeiten massgebend, was durch den grossen Massenunterschied zwischen Elektronen und Atomen begründet wird. In der vorliegenden Arbeit äussern die Verfasser ernste Bedenken gegen die allgemeine Gültigkeit dieser adiabatischen Näherung.

Für die Schrödingergleichung eines Festkörpers hat man in den gewohnten Bezeichnungen

$$(1) \quad \left(-\sum_{\mu=1}^{3n} \frac{h^2}{8\pi^2 m} \frac{\partial^2}{\partial x_\mu{}^2} - \sum_{\nu=1}^{3N} \frac{h^2}{8\pi^2 M_\nu} \frac{\partial^2}{\partial X_\nu{}^2} + V\right)\psi = E\psi,$$

wo m und x_μ die Masse und die Koordinaten der Elektronen und die M_ν und X_ν die der Gitterteilchen bedeuten. V ist der Wechselwirkungsoperator. Die adiabatische Näherung besteht in der Vernachlässigung des zweiten Gliedes auf der linken Seite von (1); dann folgt also die Wellengleichung

$$(2) \quad \left(-\frac{h^2}{8\pi^2 m} \sum_{\mu=1}^{3n} \frac{\partial^2}{\partial x_\mu{}^2} + V\right)\varphi_r = W_r(X)\varphi_r(x, X).$$

ψ kann man nach den Eigenfunktionen von (2) in eine Reihe entwickeln, erhält also

$$(3) \quad \psi(x, X) = \sum_\alpha \chi_\alpha(X)\varphi_\alpha(x, X),$$

die man in (1) einsetzen und auf diesem Wege die Funktionen χ berechnen kann. Mit Hilfe eines teilweise klassischen und teilweise quantenmechanischen Gedankenganges wird dann gezeigt, dass das in (2) schon vernachlässigte Glied (die kinetische Energie der Gitterteilchen) von erster und nicht von zweiter Ordnung ist, wie das der adiabatischen Näherung entsprechen würde. Ausserdem entstehen bei der Einsetzung von (3) in (1) Wechselwirkungsglieder die schon von dritter Ordnung sein müssten, aus den Rechnungen folgt jedoch, dass sie ebenfalls von erster Ordnung sind. Der Übergang von der adiabatischen zur statischen Näherung (bei der man annimmt, dass die Gitteratome unbeweglich an ihre Gleichgewichtslagen gebunden sind) ist dagegen theoretisch einwandfrei. Als

Schlussfolgerung ergibt sich, dass die erwähnte Näherung zur Bestimmung von Eigenwerten zwar brauchbar ist; die Heranziehung von H' zur Berechnung von Übergängen ist jedoch sehr bedenklich.

T. Neugebauer (Budapest)

RELATIVITY

See also 584, 599.

609:

Horváth, J. I. New geometrical methods of the theory of physical fields. Nuovo Cimento (10) **9** (1958), supplemento, 444–496.

The chapters of this work are titled: Relativistic theory of the electromagnetic field in uniformly moving dielectra; A model example of classical bilocal theory of physical fields; Differential structure of classical fields in generalized metrical space. The first chapter contains little new material and fails to mention the similar work of W. Gordon [Ann. Physik **72** (1923), 421–456].

R. A. Toupin (Washington, D.C.)

610:

Cattaneo, Carlo. Sui postulati comuni alla cinematica classica e alla cinematica relativistica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 526–532.

The author takes up the often discussed problem of axiomatic construction of relativistic kinematics. This paper has the advantage of shewing that relativistic kinematics are built with the aid of Galileian space and time. The first two groups of postulates are the same as those of classical kinematics. Only the third group of postulates concerning motion makes the characteristics peculiar to relativistic kinematics conspicuous.

O. Onicescu (Bucharest)

611:

Galli, Mario G. Vedute moderne circa i fondamenti delle trasformazioni di Lorentz. I. Riv. Mat. Univ. Parma **8** (1957), 161–198.

612:

Kichenassamy, S. Choix et interprétation des systèmes de coordonnées en relativité générale. Cahiers de Phys. **12** (1958), 311–316.

A space-time is here defined to have a symmetry S if its metric in a given coordinate-system is invariant under the group of global transformations which leave invariant the flat metric which corresponds to the interpretation of the coordinates. An explicit criterion is given for this in terms of the Lie derivatives of the vector fields generating the group.

{Whether the space-time has a certain symmetry appears to depend on the interpretations given to the coordinates. This is a satisfactory way of making precise statements like "r, θ, ϕ are roughly spherical polars", but it is not clear to the reviewer to what extent the property discussed is a property of the space-time, and to what extent it is only a property of the coordinate-system.}

C. W. Kilmister (London)

613:

Brumberg, V. A. The equations of motion and the coordinate conditions in the relativistic problem of N bodies. Astr. Ž. **35** (1958), 893–903. (Russian. English summary)

The Einstein-Infeld-Hoffmann (E.I.H.) method is used to derive the post-Newtonian equations of motion of N bodies without imposing any conditions on the coordinate system. A class of approximate solutions of the field equations depending on four arbitrary functions is obtained. It is then shown that all these solutions may be also obtained by applying coordinate transformations to a special solution found in the original paper of E.I.H. [Ann. of Math. (2) **39** (1938), 65–100] without transforming the positions of the bodies. It is proved by straightforward calculation that all solutions of this class lead to the same equations of motion. A conclusion is drawn that the E.I.H. method gives the same results in all coordinate systems and that other methods must be used to decide to which system these results really apply. {This conclusion, in the opinion of the reviewer, is wrong. The author has proved only that the equations of motion are not affected by coordinate transformations as long as the positions of the bodies are not transformed.}

W. Tulczyjew (Warsaw)

614:

Staniukovich, K. P. On the motion of bodies at high velocities in a weak gravitational field. Soviet Physics. Dokl. **120** (**3**) (1958), 558–561 (277–280 Dokl. Akad. Nauk SSSR).

The approximate equations of the motion of a continuous rarefied medium in a weak gravitational field are derived and applied to the motion of a gas which has been exploded with a large release of energy. The analysis shows that different trajectories are obtained depending on the initial conditions. After exploding, the entire gas may stop moving at a finite distance and then fall back to the centre or some particles may go to infinity and some can participate in a pulsating process. With very large energy release all the particles will tend to escape to infinity.

G. L. Clark (London)

615:

Blancheton, Éliane. Sur le tenseur impulsion-énergie d'un champ électromagnétique. C. R. Acad. Sci. Paris **248** (1959), 372–374.

The paper begins with a simple proof of a theorem due to Rainich [*Mathematics of relativity*, Wiley, New York, 1950; MR **13**, 78]. Cf. also Misner and Wheeler [Ann. Physics. **2** (1957), 525–603; MR **19**, 1237]. Necessary and sufficient conditions that a tensor $E^{\alpha\beta}$ be expressible in the form

$$E^{\alpha\beta} = \tfrac{1}{4} g^{\alpha\beta} H^{\lambda\mu} H_{\lambda\mu} - H^{\alpha\lambda} H_{\lambda}{}^{\beta},$$

where $H^{\alpha\beta} = -H^{\beta\alpha}$ and $g^{\alpha\beta}$ is the metric tensor, are

$$E^{\alpha\beta} = E^{\beta\alpha}; \quad g_{\lambda\mu} E^{\lambda\mu} = 0; \quad g_{\lambda\mu} E^{\alpha\lambda} E^{\mu\beta} = \sigma g^{\alpha\beta}, \; \sigma \geqq 0.$$

If $\sigma = 0$, the corresponding electromagnetic field $H^{\alpha\beta}$ is called singular. If H is singular, then E has the form $E^{\alpha\beta} = \tau L^{\alpha} L^{\beta}$ where L is a null vector. The author then considers an expansion of the solutions $(g_{\alpha\beta}, E^{\alpha\beta})$ of the general relativity field equations $S^{\alpha\beta} = \chi E^{\alpha\beta}$ about a solution $(g_{\alpha\beta}, {}_0E^{\alpha\beta})$, where ${}_0E^{\alpha\beta}$ is the energy tensor of a singular field.

R. A. Toupin (Washington, D.C.)

616:

Udeschini, Paolo. Sviluppi dell'ultima teoria unitaria di Einstein. Rend. Sem. Mat. Fis. Milano **27** (1957), 50–74.

This is an excellent survey of the development of the latest Einstein unified field theory. The author discusses all pertinent contributions to this theory and their interrelations. He starts with both systems of field equations ("weak" and "strong" system) and deals with the determination of the unified connection, considering in detail first and second approximation of solutions as well as the exact spherically symmetric solutions and the Cauchy problem. Among other items the reader will find an account of equations of motion and of the momentum energy tensor. The paper ends with a survey of theories modifying the Einstein theory. Besides the original sketch of Einstein the papers of the following authors are discussed: Finzi, Schrödinger, Kaufman, Tonnelat, Hlavatý, Lichnerowicz, Udeschini, Papapetrou, Wyman, Bonnor, Vaidya, de Simoni, Infeld, Schild, Callaway, Kursunogle, and Böse. The paper discloses a complete mastery of all angles and details of Einstein's Unified Theory.

V. Hlavatý (Bloomington, Ind.)

617:

Madhava Rao, B. S. Physical applications of the Lorentz group. Proc. Indian Acad. Sci. Sect. A. **47** (1958), 105–115.

"The role of special relativity, based on the restricted Lorentz group L_4 consisting of rotations, in space-time, in the earlier development of quantum theory, in the connection between spin and statistics established by Pauli, and in Dirac's theory of the electron is briefly indicated. The theory of undors is pointed out as an example where, besides invariance under L_4, invariance under charge-conjugation and space-time reflections also is satisfied.

The Pauli-Luders theorem taking into account the interactions between elementary particles, and the subsequent work of Lee and Yang relating to parity, charge-conjugation and time-reversal are explained, and cited as examples of general representations of the Lorentz group." (From author's summary)

K. Yano (Tokyo)

618:

Hämeen-Anttila, K. A. Eine Theorie der Gravitation und des Elektromagnetismus. Soc. Sci. Fenn. Comment. Phys.-Math. **20** (1957), no. 4, 50 pp.

The special relativity theory gives a complete unified theory for electric and magnetic forces. The invariance of the Maxwell equations under Lorentz transformations unifies these two forces and permits to represent them as a simple entity in the four-dimensional Minkowski space.

A five-dimensional Riemannian space satisfying certain conditions has been used to get a physical theory in which the gravitational field and the electromagnetic field are unified and are represented by a simple geometric object.

In the paper under review, the author tries to get a new theory unifying gravitation and electromagnetism by an invariance principle which will be derived from the structure of classical physics and which connects the mass and the charge of a particle in the way which was explained in the above.

K. Yano (Tokyo)

619:

Kilmister, C. W. Eddington's statistical theory. IV. The nuclear potential. Rend. Circ. Mat. Palermo (2) **6** (1957), 311–324.

[For parts I–III, see E. W. Bastin and Kilmister, same Rend **5** (1956), 187–203; MR **18**, 782; Kilmister, ibid. **6** (1957), 33–50; MR **20** #1582; Kilmister and B. O. J. Tupper, ibid. **6** (1957), 117–140; MR **20** #1583.]

The author attempts to give a more satisfactory discussion of section five of Eddington's book, *Fundamental theory* [University Press, Cambridge, 1946; MR **11**, 144]. He states that the potential function entering into the Schrödinger non-relationistic equation in Euclidean space for the scattering of a particle of charge e by a scatterer with equal charge and mass is not e^2/r but is $e^2r+\phi_1$. The function ϕ_1 is supposed to arise because the solutions of the Maxwell equations in flat space-time are said to involve calculated distances whereas the Schrödinger equation refers to measured distances. Although a cosmological model of the universe is involved, the equations treated are always referred to a flat space-time. An approximate expression for the function ϕ_1 is obtained and remarks are made concerning a relationistic theory.

A. H. Taub (Urbana, Ill.)

620:

Newman, D. J. Structure theory. Proc. Roy. Irish Acad. Sect. A **59** (1958), 29–47.

An attempt to develop a scheme of concepts to describe algebraically procedures of measurement, in order to investigate the hypothesis that a new postulate of impotence "That we cannot discover whether the orderliness in our physical picture arises in our minds or from properties of the universe" is true. Basic concepts: scale, entity, observe (a real number), graduation (of scale), structural coefficient (an algebraic symbol representing a scale). Axioms are laid down to correspond (as far as the abstract nature of the scheme allows) to length measurement. The algebraic expression of the scheme (quaternions) arises from assumptions of relations between structural coefficients, corresponding to properties of length measurement.

Extensions to the direct product algebra are made; this arises in the joint measurements of space and time in relativity theory. There is also some discussion of particles and fields, but in the opinion of the reviewer the later part of the paper is less well thought out. The first part appears to be an important new development.

C. W. Kilmister (London)

ASTRONOMY

See also 276, 619.

621:

Meffroy, Jean. Sur un cas d'élimination du terme séculaire pur introduit dans la perturbation du troisième ordre des grands axes par le coefficient d'argument nul de la fonction perturbatrice. C. R. Acad. Sci. Paris **247** (1958), 863–865.

Consider a three-body problem with eccentricities of the perturbed and the perturbative planets equal to zero. The corresponding inclinations, being by assumption different

from zero, may assume small and large values. The author shows that in such a case it is possible to eliminate the purely secular term introduced in the third order perturbation of the major axes by the coefficient $N_{0,0}$ of the perturbation function. This fact shows that the eccentricities of the two planets play a prevailing role in the formation of this secular term.

E. Leimanis (Vancouver, B.C.)

622a:

Vernić, Radovan. Kritische Betrachtungen über die Zusammenstösse im Mehrkörperproblem. Rad. Jugoslav. Akad. Znan. Umjet. **314**. Odjel Mat. Fiz. Tehn. Nauke 7, 5–85 (1957). (Serbo-Croatian. German summary)

622b:

Vernić, Radovan. Die Lösung des Mehrkorperproblems. Rad. Jugoslav. Akad. Znan. Umjet. **314**. Odjel Mat. Fiz. Tehn. Nauke **7**, 111–186 (1957). (1 insert) (Serbo-Croatian. German summary)

In a series of papers [*Diskussion der Sundmanschen Lösung des Dreikorperproblems*, Jugoslav. Akad. Znanosti i. Umjetnosti, Zagreb, 1954; Jugoslav. Acad. Znan. Umjet. Rasprave. Odj. Mat. Fiz. Tehn. Nauke **1** (1952), 81–123; Rad. Jugoslav Acad. Znan. Umjet. **302**. Odjel. Mat. Fiz. Tehn. Nauke **6**, 47–73 (1955); Hrvatsko Prirodoslovno Društvo. Glasnik. Mat.-Fiz. Astr. Ser. II 8 (1953), 247–266; **9** (1954), 3–13; MR **16**, 867, 529; **17**, 94; **16**, 181, 868] the author claims to have extended the Sundman solution of the three-body problem to the complete solution of this problem. Several writers, however, have objected to this claim by pointing out several inconsistencies and contradictions to other well-known works [cf. J. Lense, Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 237–242 (1957); MR **19**, 257 and G. A. Merman, Byull. Inst. Teor. Astr. **6** (1956), 408–415; MR **19**, 227].

The aim of the present two papers is to carry over the above claims to the general $n(\geq 4)$-body problem. For this purpose the author uses historical and critical arguments which in the reviewer's opinion, however, suffer from the same deficiencies as mentioned above for the three-body case. The second paper is mainly concerned with details of the exposition.

The section headings of both papers are the following: 1. Existence of solutions. 1.1 Collisions as the only singularities. 1.2 The limit relations. 2. Construction of solutions. 2.1 Local regularization and uniformization. 2.2 Total regularization and uniformization.

E. Leimanis (Vancouver, B.C.)

GEOPHYSICS

See also 506, 565.

623:

Duwalo, George; and Jacobs, J. A. Effects of a liquid core on the propagation of seismic waves. Canad. J. Phys. **37** (1959), 109–128.

In problems dealing with the propagation of seismic waves the earth can be idealized by a homogeneous spherical liquid core ($r<a$), surrounded by a homogeneous solid mantle. The elastic disturbances are of longitudinal type (compressional waves, or P waves) or of a transverse type (shear waves, or S waves). An arbitrary solution $s(x, y, z)$ for the displacements can be expressed in terms of three scalars ϕ, ψ and χ; in the time-harmonic case ϕ satisfies the longitudinal Helmholtz equation $(\Delta+k_p^2)\phi=0$, ψ and χ the other transverse Helmholtz equation $(\Delta+k_s^2)(\psi, \chi)=0$. The field due to a point source (epicentre) can therefore be expanded in modes consisting of the product of a spherical Bessel or Hankel function, and a Legendre function $Pn(\cos\vartheta)$. The propagation problem is very similar to that of radio waves diffracted around the earth; in the latter problem the earth and its surrounding atmosphere (both assumed as homogeneous) correspond to the core and the mantle respectively in the seismic problem. However, the boundary condition along the spherical interface between mantle and core involves a coupling of the contributions depending on ϕ, ψ and χ which has no equivalent in the radio problem. Therefore, a source of pure P waves in the mantle will produce additional S waves due to the interaction along the interface. In spite of this complication the authors could treat the seismic problem by the same methods as applied by Van der Pol and Bremmer [Philos. Mag. (7) **24** (1937), 141–176, 825–864] for the diffraction of electromagnetic waves around a sphere. In fact, the expansion for each of the three scalars is split formally (leaving out of consideration difficulties regarding the convergence of the new expansions) into further series each of which represents a contribution that has suffered a special number of reflections and refractions along the interface. The assumption of a liquid core excludes the presence of shear waves in the latter; on the other hand, the refraction of P waves emerging from the core towards the mantle depends on two transmission coefficients, since the P wave leaving the core penetrates into the mantle partly as another P wave, partly as a S wave. An application of the Watson transformation leads, just as in the radio case, to a residue series which converges rapidly in the shadow region; the various transmission and reflection coefficients constitute the physical parameters of this series. The shadow region is limited at one side by the ray trajectory tangenty the core. The simplest case dealing with a source of pure shear waves is discussed first; it corresponds to the case of a perfectly conducting earth in the radio problem.

H. Bremmer (Eindhoven)

624:

Helfenstein, H. G. Critical curves in seismic exploration. Math. Mag. **31** (1957/58), 85–91.

The basic problem is to determine an underground surface of discontinuity, D, from seismic reflections. If the wave travel distance is known as a function of the position of the receiver, D may be determined as the envelope of a family of ellipsoids. The author shows that the envelope is undetermined and the method fails if the recording points are confined to certain straight lines or hyperbolas on the surface which pass through the shot point.

E. Pinney (Berkeley, Calif.)

625:

Bragard, L. Sur la condensation ellipsoïdale du relief topographique. Bull. Soc. Roy. Sci. Liège **27** (1958), 185–188.

Es wird zunächst die Dichte der Flächenbelegung auf einem Ellipsoid bestimmt, welche das selbe Potential besitzt wie ein vorgegebener Massenpunkt. Hierbei wird

von Kugelfunktions-Entwicklungen Gebrauch gemacht, deren Größe etwa der Genauigkeit entsprechen soll, mit der die Schwereanomalien, relativ genommen, bestimmt werden können. Dann werden diese Dichtefunktionen (je für den Innen- und den Außenraum) in die Integralformel für die Geoid-Undulationen eingesetzt, woraus man den Anteil der topographischen Massen an den Geoid-Undulationen durch einen Summationsprozeß gewinnen kann. *H. Wolf* (Bonn)

626:

Hultqvist, Bengt. The spherical harmonic development of the geomagnetic field, epoch 1945, transformed into rectangular geomagnetic coordinate systems. Ark. Geofys. **3** (1959), 53–61.

"The spherical harmonic development of the earth's magnetic field, for epoch 1945, has been transformed into rectangular coordinate systems with the z-axis in the direction opposite to that of the earth's dipole and with the x-axis in those 36 geomagnetic meridian planes having a geomagnetic longitude of 0°, 10°, 20° $\cdots$ 350° respectively." (Author's summary)

E. Kogbetliantz (New York, N.Y.)

627:

Hultqvist, Bengt. The geomagnetic field lines in higher approximation. Ark. Geofys. **3** (1959), 63–77.

"The deviation at the earth's surface of the geomagnetic field lines, in an approximation including the five first spherical harmonic terms, from the dipole lines has been calculated with a perturbation method, on the basis of Vestine's et al. analysis of the geomagnetic field for epoch 1945, and the result is discussed with regard to some problems in geomagnetism, aurora, cosmic rays and whistlers." (Author's summary)

E. Kogbetliantz (New York, N.Y.)

628:

Agostinelli, Cataldo. Ulteriori considerazioni sul movimento di un ghiacciaio e sulla determinazione del profilo di una sua sezione retta. Boll. Un. Mat. Ital. (3) **13** (1958), 327–334. (English summary)

"We establish the differential equation of the profile in the right section of the glacier, with the law of the velocity distribution on the free surface. We examine in detail the case in which this law is parabolical."

Author's summary

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 314, 315, 317, 378, 393, 410, 446, 639.

629:

Arrow, Kenneth J.; and Hurwicz, Leonid. On the stability of the competitive equilibrium. I. Econometrica **26** (1958), 522–552.

The classical system of supply-and-demand equations for a competitive market can be regarded as defining the stationary solution of a more general system of dynamic equations. The question is whether the stationary solution exists and is stable. In the present article this question has been investigated for a particular form of the dynamized model. *T. Haavelmo* (Oslo)

630:

Dantzig, G. B.; Fulkerson, D. R.; and Johnson, S. M. On a linear-programming, combinatorial approach to the traveling-salesman problem. Operations Res. **7** (1959), 58–66.

The authors present with greater elaboration their linear programming solution [refer to the review of Dantzig, Fulkerson, and Johnson, J. Operations Res. Soc. Amer. **2** (1954), 393–410; MR **17**, 58] to the shortest route taken between ten cities given by L. L. Barachet [Operations Res. **5** (1957), 841–845]. Six figures and two computation tables are given to illustrate the steps which begin by computing simplex multipliers called potentials which are then used for introducing new vectors into the basis in the iterations of the simplex process. A brief and interesting discussion of the practicality of the method is finally given which encourages the use of the method because of successful experience with it.

T. L. Saaty (New York, N.Y.)

631:

Kron, Gabriel. Multiple substitution of basic vectors in linear programming. I, II. Matrix Tensor Quart. **7** (1956), 3–11, 48–50.

The author points out that inserting several vectors in the basis simultaneously in a simplex calculation may save time, especially if one is lucky in choosing vectors to enter and leave so that the new basis is feasible.

A. J. Hoffman (New York, N.Y.)

632:

Kelley, J. E., Jr. A threshold method for linear programming. Naval Res. Logist. Quart. **4** (1957), 35–45.

Gerstenhaber (unpublished) has suggested a so-called "threshold method" for solving the transportation problem. This paper extends the method to the general linear programming problem. The idea is to concoct a vector function (depending on a positive parameter r) which maps a subset of vectors in the dual space onto a subset of vectors in the primal space in such a way that, if an image vector is feasible, then the associated value of the objective function differs from the optimum by at most r. The author lists postulates for such a mapping and exhibits a specific mapping satisfying the postulates. The computing procedure is still in the experimental stage, and no experience is reported.

A. J. Hoffman (New York, N.Y.)

633:

★Goldman, A. J.; and Tucker, A. W. Theory of linear programming. Linear inequalities and related systems, pp. 53–97. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N.J., 1956. $5.00.

This paper, which is ostensibly expository although it contains much new material, discusses systematically the basic properties of dual linear programs: $Ax \leqq b$, $x \geqq 0$, $\max(c, x)$; $y'A \geqq c'$, $y \geqq 0$, $\min(b, y)$. Parts 1 and 2 develop the duality theorem in a straightforward way, concluding with the explicit statement (Corollaries 2A and 2B) of the results characterizing the circumstances when pairs of dual constraints are satisfied as equations or inequalities. More on this question is given in part 6. These results are occasionally stated rather sloppily (and incorrectly) elsewhere.

Part 3 treats systems with mixed constraints of equations and inequalities, and some variables not required to be non-negative. Although well-known simple tricks suffice to recast the problems in canonical form, the authors do better by exhibiting two elimination procedures either of which succeeds either in exhibiting the problem pair as unfeasible, or bringing the problem pair into an equivalent form of smaller size.

Part 4 discusses the connection between pairs of dual programs and matrix games. Part 5 treats the connection with Lagrange multipliers. Theorem: If

$$L(x, y) = (c, x)+(b, y)-(y, Ax),$$

then $x^0 \geqq 0$ and $y^0 \geqq 0$ are optimal vectors if and only if

$$L(x, y^0) \leq L(x^0, y^0) \leq L(x^0, y)$$

for all $x \geqq 0$, $y \geqq 0$.

Part 7 gives a description of the set of all optimal solutions to either primal or dual as the sum of the convex hull of the optimal extreme feasible solutions and a certain convex cone. Finally, the principal result of Section 8 is a proof that, given any preassigned polyhedral convex sets in primal and dual space respectively, they can be made optimal sets of some pair of dual programs.

A. J. Hoffman (New York, N.Y.)

634:

Brackney, Howard. The dynamics of military combat. Operations Res. 7 (1959), 30–44.

The author gives simple formulations for combat loss of two opposite forces as done by Lanchester and relates this to the notion of search during combat by means of equations which express the dependence of search periods upon the target densities in the searched areas. Finally the idea of dependence on the method of use of weapons in combat rather than their superiority is introduced. Having derived nine different combat configurations, figures are then given to show that only three general types of solutions arise. Indicated are situations in which the superior force regardless of what the enemy does (1) assaults; (2) remains in defensive action under cover in a position of constant area; (3) remains in defensive action under cover with a constant density of force over its position. Some suggestions for actual combat are then given relating the ideas to modern warfare.

T. L. Saaty (New York, N.Y.)

635:

★von Neumann, John. On the theory of games of strategy. Contributions to the theory of games, Vol. IV, pp. 13–42. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

Translation by Mrs. Sonya Burgmann of "Zur Theorie der Gesellschaftsspiele" [Math. Ann. **100** (1928), 295–320].

636:

★Kemeny, John G.; and Thompson, Gerald L. The effect of psychological attitudes on the outcomes of games. Contributions to the theory of games, vol. 3, pp. 273–298. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N.J., 1957. $5.00.

Psychological "attitudes" of various sorts are modeled in game theory by attributing different "utility functions" to the players. A utility function is a real non-constant monotone increasing function. A utility function is "strategy-preserving" if for all games $\|g_{ij}\|$ and for all real h the set of optimal strategies is the same for all the games with the matrices $\|f(g_{ij}+h)\|$. If a player has a strategy-preserving utility function f, then his play will be independent of his current fortune (estate). Part of the paper shows that all such functions must be linear or exponential plus constant with $f'(x)>0$ in the 2-person 0-sum game. Similar definitions and results are obtained for the n-person equilibrium-point situation and for stochastic games.

Now by imposing various utility functions it is suggested that one may obtain patterns of play characteristic of "fair", "reckless", "cautious" (the linear and exponential cases); "rich man", "poor man", "common", "winning" and "desperate" attitudes in the sense that there will be analogous distortions of the utilities of large and small wins and losses in appropriate directions. A number of suggestive examples are given, e.g., lotteries, insurance and simple military problems.

The next section shows how the different utility functions can influence the course of (finite) sequential games in which one player has the option of terminating the game. If the player adds his own special premium to winning to the regular matrix payoff (the "winning" attitude) then his strategy will depend on his current fortune (and he will terminate the game as soon as he is ahead or irretrievably lost).

In a final section precise definitions are assigned to the "reckless", "cautious" and "ordinary" attitudes and theorems are obtained concerning the forms of the utility functions that produce the prescribed behaviors.

M. L. Minsky (Cambridge, Mass.)

BIOLOGY AND SOCIOLOGY

See also 448, 636.

637:

Farley, B. G.; and Clark, W. A. Simulation of self-organizing systems by digital computer. Trans. I.R.E. **PGIT-4** (1954), 76–84.

The systems examined here consist of randomly interconnected nets of idealized "neuron"-like elements. It is desired to have the nets learn to respond in specified ways to different classes of inputs. This is done through use of a pair of "modifying" operators which are chosen according to whether or not the behavior has just been observed to change in a favorable direction. If the change is considered favorable, the chosen modifier increases the potency of those internal connections which have recently been active. The empirical results show that this kind of operation, for which the reviewer suggests the term "structural reinforcement", does indeed yield the desired kind of learning.

The important feature of such a system is that application of the modifier is based solely on the behavior, and not on detailed examination of what has happened within the net, so that in principle such a machine could solve problems in ways not anticipated by the programmer. Computations were performed on the high-speed experimental machine, MTC, of the M.I.T. Lincoln Laboratory.

Description of further experiments on this system, concerning pattern recognition and stimulus generalization are published in the paper reviewed below.

M. L. Minsky (Cambridge, Mass.)

638:

★Clark, W. A.; and Farley, B. G. Generalization of pattern recognition in a self-organizing system. Proceedings of the Western Joint Computer Conference, held March 1–3, 1955 in Los Angeles, California, pp. 86–91. Institute of Radio Engineers, New York, 1955. 132 pp. $3.00.

An improved exposition is given of the report reviewed above for a learning system based on randomly interconnected neuron-like elements. Further experiments are discussed concerning the abilities of this system to perform stimulus-generalization and pattern-recognition. The net is first organized with two distinct stimuli as described in the earlier experiment. The learning system is then inactivated and the response properties are studied with regard to stimuli similar, in the sense of overlap, to the organizing stimuli. Learning does turn out to spread well in accord with this notion of similarity, and the results are discussed in connection with possible application to a visual pattern recognition experiment.

M. L. Minsky (Cambridge, Mass.)

639:

Kochen, Manfred; and Galanter, Eugene H. The acquisition and utilization of information in problem solving and thinking. Information and Control **1** (1958), 267–288.

A discussion of the distinction between actions directed at acquisition of information and actions exploiting for gain already acquired information. A subject is presented with a periodic binary sequence and for each term may either: (1) pay for being told what its value is, (2) bet on its value (without, however, being told the outcome until the very end), or (3) bet on the whole sequence, different payoffs being assigned to the cases. Several kinds of strategies are discussed informally. There is then some discussion of ways in which one might program a machine to make such decisions, e.g., the machine might remember the n-games which have occurred and use them until they fail (then using longer ones). This scheme is related to the "inductive inference" schemes proposed by Solomonoff [IRE Convention Record **5** (1957), 56–62]. Some preliminary experiments on human subjects are reported.

M. L. Minsky (Cambridge, Mass.)

INFORMATION AND COMMUNICATION THEORY

See also 382, 446, 448, 449.

640:

★Perry, J. W.; and Kent, Allen. Tools for machine literature searching: semantic code dictionary, equipment, procedures. With the semantic code dictionary under the general editorship of John L. Melton. Library Science and Documentation. Vol. 1. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1958. xviii+972 pp. $27.50.

One of the pioneer and productive groups in the development of new equipment and techniques of documentation have capped the long list of their contributions with the publication of a "semantic code" dictionary and the necessary explanatory material to permit its exploitation in other hands. The cost of the book can scarcely equal one one-thousandth part of the cost of its composition. In their capacity as "literature searchers", mathematicians will watch with sympathy the efforts of this and other groups.

Mathematical interest centers more directly in the formal properties of the system. The authors (page 20) assert "it is no exaggeration to say that the basic logic of class definition provides the link between machine operation and recorded subject matter." But later (page 29) "... the theory of class definition does not provide guidance in arriving at decisions as to the selection and definition of characteristics."

The authors do not seem to realize that they provide no abstract theory of the retrieval process, nor that one would help them in their labors though they do provide a description of a particular retrieval system which others may use for the purpose of constructing such an abstract theory.

C. J. Maloney (Frederick, Md.)

SERVOMECHANISMS AND CONTROL

641:

Mitropol'skiĭ, Yu. A. Inner resonance in non-linear oscillating systems. Kiïv. Derž. Univ. Nauk Zap. **16** (1957), no. 2 = Kiev. Gos. Univ. Mat. Sb. **9** (1957), 53–61. (Russian)

The author considers nonlinear systems characterized by differential equations of the form

$$(1) \qquad Z(p)x = \varepsilon F(\tau, x)$$

where $p=d/dt$, $\tau=\varepsilon t$, ε is a small parameter, $F(\tau, x)$ is a given functional and $Z(p)=\sum_{n=1}^{N} a_n(\tau)p^n$ is expressible as

$$Z(p) = [p^2+\omega^2(\tau)]Q_1(p)+\varepsilon R(\tau, p),$$

Q_1 and R being polynomials in p. Under not clearly stated assumptions, the author obtains a set of differential equations satisfied by the successive terms in a power series expansion of a solution of (1), and analyses in detail a simple example. The paper contains a number of misprints and ambiguities.

L. A. Zadeh (New York, N.Y.)

642:

Helm, H. A. The Z transformation. Bell System Tech. J. **38** (1959), 177–196.

The author defines the z-transform of a time-function $f(t)$ by the Laplace-Stieltjes integral

$$F(z) = \int_0^\infty f(t)z^{-t/T}d\alpha(t),$$

where T is the sampling interval, z is a complex variable and $\alpha(t)$ is a "staircase" function taking the values $\alpha(t)=k$ for $kT<t\leq(k+1)T$ $(k=\cdots, -1, 0, 1, 2, \cdots)$. In this way, the use of delta-functions is avoided and, in a few instances, a higher level of rigor is attained. However, in establishing the various basic properties of z-transforms

the author relies for the most part on the conventional expression $F(z)=\sum_{n=0}^{\infty} f(nT)z^{-n}$.

L. A. [illegible] (New York, N.Y.)

643:

Šefl, O. Filtering of noise in optimalizing control. Sci. Sinica **7** (1958), 1144–1150.

The author considers a decision-detection process in which it is permissible to use linear filters. The analytic problem can then be reduced to that of minimizing certain quadratic expressions.

R. Bellman (Santa Monica, Calif.)

644:

Šefl, O. On stability of a randomized linear system. Sci. Sinica **7** (1958), 1027–1034.

The author considers linear systems with random coefficients of the form

$$\frac{dx}{dt} = (A+B(t))x, \quad x(0) = c,$$

where $B(t)$ is a matrix whose elements are stationary random functions. Using the techniques available in the deterministic case [cf. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York–Toronto –London, 1953; MR **15**, 794], he obtains various estimates for the probability of stability.

R. Bellman (Santa Monica, Calif.)

HISTORY AND BIOGRAPHY

See 1, 459.

MISCELLANEOUS

See 386, 555.

AUTHOR INDEX—*continued*